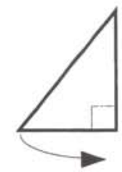
Conic Sections Lesson #1:
Introducing Conic Sections

The Double-Napped Cone



A **cone** is a solid which can be generated by rotating a right angled triangle about one of its legs.

A **double-napped cone** is formed when the vertices of two cones are placed together. For example, we can make a double-napped cone by:

- taking two water cups shaped as cones and placing their tips against each other, or,
- placing a lecture pointer stick between your thumb and finger, holding it vertically in front of you, and then rotating it so that the top of the pointer and the bottom of the pointer form circles.

In general, a double-napped cone is produced by rotating an oblique line (called the generator) about an axis.

If the line is parallel to the generator, a cylinder is formed. This case will be discussed later.

A double-napped cone consists of the following parts: **generator**, **upper nappe**, **lower nappe**, **vertex**, **axis of symmetry** (called the **central axis**), **generator angle**, and **vertex angle**.

Label the following on the diagram.

a) the generator

b) upper nappe

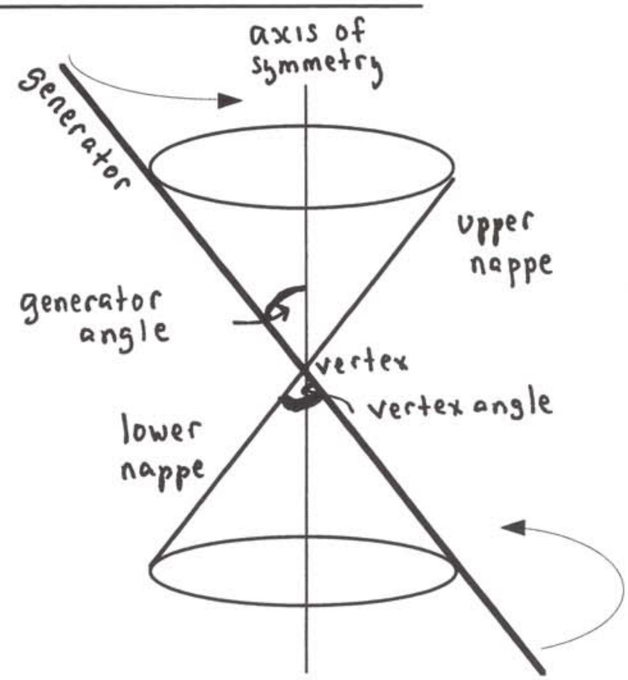
c) lower nappe

d) vertex

e) axis of symmetry

f) generator angle

g) vertex angle



Conic Sections

Conic sections are two dimensional figures which can be formed by a plane slicing a double-napped cone (or a cylinder).

Much of the work in this area of mathematics was discovered by the Greek mathematician Apollonius in about 200 B.C. He discovered that the intersection of a plane and a double-napped cone could result in one of four different conic sections, called the **primary conic sections**, according to the angle of intersection.

The Primary Conics Generated from a Double-Napped Cone

The angle of intersection between a cutting plane and a cone is defined as the angle between the central axis and the cutting plane.
If we take a plane and cut a double-napped cone at different angles to the axis and not through the vertex, we generate the **primary conics** illustrated below.

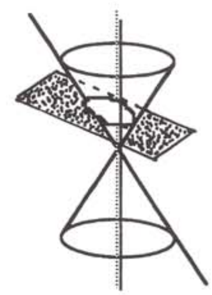
Circle If the plane cuts the cone such that the plane is perpendicular to the central axis, then the primary conic generated is a circle.

Ellipse If the plane cuts the cone such that

- the plane is neither perpendicular nor parallel to the axis,

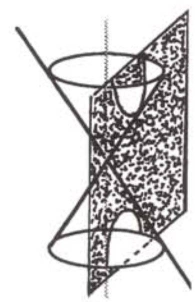
  and
- the angle of intersection is greater than the generator angle,

then the primary conic generated is an ellipse.



Parabola If the plane cuts the cone such that the plane is parallel to the generator, then the primary conic generated is a parabola.

Hyperbola If the plane cuts the cone such that the angle of intersection is less than the generator angle, then the primary conic generated is a hyperbola. In this case, the cutting plane intersects both nappes of the cone.



Describing a Primary Conic by the Cutting Plane and Central Axis

Conic sections can be determined from the angle formed by the cutting plane and the central axis of a double napped cone.

Class Ex. #2

A plane intersects a double-napped cone. The primary conic produced if the plane does not pass through the vertex and cuts the cone;

a) perpendicular to the axis is a circle.

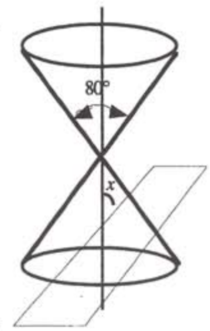
b) at an angle equal to the generator angle is a parabola.

c) parallel to the axis is a hyperbola

d) at an angle greater than the generator angle is an ellipse.

e) at an angle less than the generator angle is a hyperbola.

Class Ex. #3



The vertex angle of a double-napped cone is 80°. The angle between the cutting plane and the central axis is x. Determine the value, or range of values, of x which would generate

a) a circle $x = 90°$

b) a parabola $x = 40°$

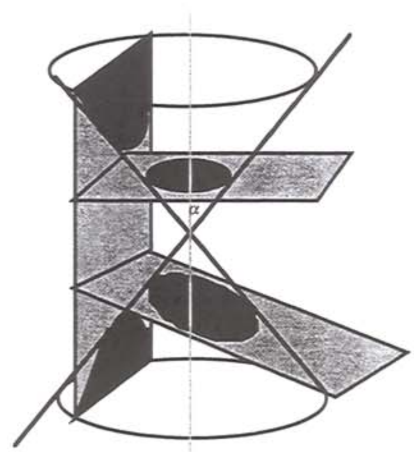
c) an ellipse $40° < x < 90°$

d) a hyperbola $0° \leq x < 40°$

Class Ex. #4

In the diagram α is the angle between the axis and the generator. Let θ be the angle between the cutting plane and the axis.

For each of the primary conics, describe the relationship between α and θ.



circle $\theta = 90°$

parabola $\theta = \alpha$

ellipse $\alpha < \theta < 90°$

hyperbola $0° \leq \theta < \alpha$

Class Ex. #5

Consider a cutting plane intersecting a double-napped cone at an angle just greater than the generator angle.

a) Which conic section is generated?

ellipse

b) What happens to the shape of this conic as the angle between the cutting plane and the axis increases towards 90°?

ellipse becomes more and more circular in shape

c) What happens at 90°?

ellipse becomes a circle

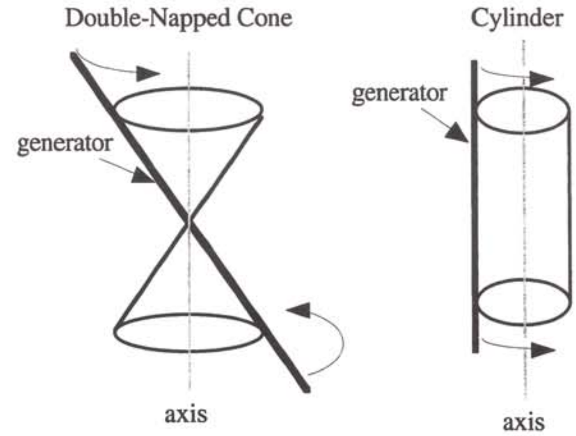
Note

- As the cutting plane gets closer and closer to 90°, the ellipse gets more and more circular until the **limiting case of the ellipse is the circle**.

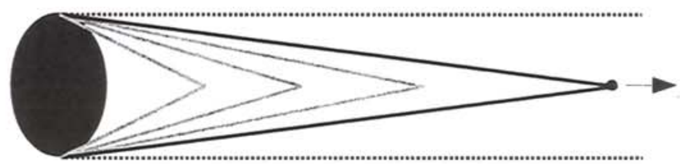
The Primary Conics Generated from a Cylinder

A double-napped cone is produced by rotating an oblique line (called the generator) about an axis.

A cylinder is produced if the generator is parallel to the axis.



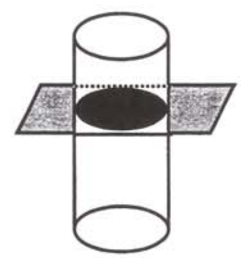
A cylinder can also be regarded as the limiting case of a cone where the vertex is stretched to infinity.



Two of the primary conics (the circle and the ellipse) can be modelled from a cylinder.

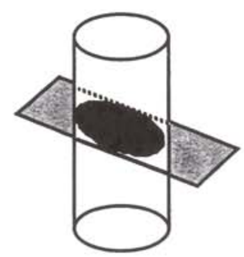
Circle

When a plane cuts through a cylinder perpendicular to the axis, a circle is produced.



Ellipse

When a plane cuts through a cylinder neither perpendicular nor parallel to the axis, an ellipse is produced.



Warm-Up

Consider the situation where a plane cuts a cone perpendicular to the axis.

a) Which primary conic section is generated? circle

b) What happens to this conic section as the cutting plane gets closer and closer to the vertex? the circle gets smaller as the radius decreases

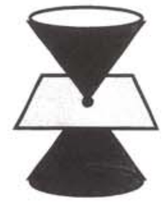
c) What happens to this conic section when the cutting plane passes through the vertex?

it becomes a point

The Degenerate Conics from a Double-Napped Cone

In the Warm-Up we saw that as the cutting plane moves closer and closer to the vertex, the circle gets smaller and smaller until eventually, when it passes through the vertex, the circle **degenerates** into a point. The degenerate conics are listed below.

Point

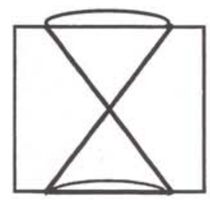


When a plane cuts a cone at an angle greater than the generator angle and passes through the vertex, a point results.
The **point** is a degenerate conic of a **circle** or an **ellipse**.

A Single Line

When a plane parallel to the generator passes through the vertex, a single line results.
The **single line** is a degenerate conic of a **parabola**.

Two Intersecting Lines



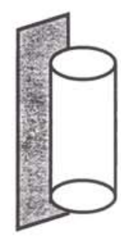
When a plane cuts through the vertex and through both nappes of the cone, two intersecting lines result.
Two intersecting lines is the degenerate conic of a **hyperbola**.

- Notice that the degenerate conics formed from a **double-napped cone** only occur when the **cutting plane passes through the vertex**.

The Degenerate Conics from a Cylinder

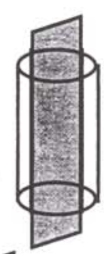
Consider the intersection of a cylinder and a cutting plane parallel to the generator.

A Single Line



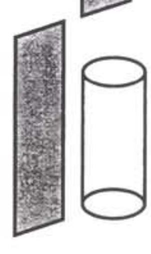
When the plane is **tangent** to the curved surface of a cylinder, a **single line** is produced.

Two Parallel Lines



When the plane cuts through a cylinder parallel to the axis, the result is **two parallel lines**.

No Graph (or No Locus)



When the plane does not intersect a cylinder, then there is no graph (or no locus).

- Since the cutting plane is parallel to the generator, the degenerate conics of a single line, two parallel lines, and no graph are regarded as degenerates of a parabola.

Complete Assignment Questions #1 - #20

Assignment

1. List: **a)** all the primary conic sections. **b)** all the degenerate conics.

circle ellipse
parabola hyperbola

point, single line, two parallel lines
two intersecting lines, no locus

2. Consider a double-napped cone whose central axis is vertical. Name the conic section produced in each case:
   a) when a plane intersects the cone parallel to the base of the cone and does not pass through the vertex.
   circle
   b) when a plane intersects both nappes of a double-napped cone and does not pass through the vertex.
   hyperbola
   c) when a plane intersects one nappe of a double-napped cone and is parallel to the generator.
   parabola
   d) when a plane intersects one nappe of a double-napped cone and is not parallel to the generator.
   ellipse
   e) when a horizontal plane passes through the vertex. point
   f) when an oblique plane passes through the vertex and contains the generator of a double-napped cone.
   line
   g) when a plane is parallel to the axis of a double-napped cone and does not pass through the vertex.
   hyperbola
   h) when a plane is parallel to the axis of a double-napped cone and passes through the vertex.
   two intersecting lines
   i) when a plane intersects a double-napped cone perpendicular to the axis of the cone and does not pass through the vertex.
   circle

3. How would you cut a double-napped cone to produce:
   a) a circle
   perpendicular to the axis and not through the vertex.
   b) an ellipse
   through one nappe at an angle greater than the generator angle and less than 90° and not through the vertex.
   c) a parabola
   parallel to the generator and not through the vertex.
   d) a hyperbola
   through both nappes at an angle between 0° and the generator angle and not through the vertex (incl. 0°)
   e) a single line
   parallel to the generator and through the vertex.
   f) two intersecting lines
   through both nappes at an angle between 0° and the generator angle and through the vertex.
   g) a point
   through the vertex at an angle between the generator angle and 90° (incl. 90°)

4. Which conic section(s) cannot be generated unless you consider the intersection of a plane and a cylinder?

two parallel lines

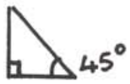
5. Which conic section(s) cannot be generated unless the plane intersects <u>both</u> nappes of a double-napped cone?

hyperbola, two intersecting lines

6. Which conics (including degenerates) can be generated by the intersection of a plane and a cylinder?

circle, ellipse, single line, two parallel lines, no locus

7. A cone is formed by rotating a right isosceles triangle about one of the equal sides. Which primary conic section is produced if this cone is intersected by a plane at 45° to the axis of the cone?



the cutting plane is parallel to the generator

parabola

8. A flashlight is pointed at a wall so that the angle between the beam and the wall is 65°.

a) Which conic section is produced? ellipse

b) How would you adjust the angle of the beam to produce a circle on the wall?

adjust the angle to 90°

9. a) A plane (not through the vertex) intersects a double-napped cone perpendicular to the axis of the cone. This plane is slowly rotated through an angle of 90° forming different conics as it rotates. In which order are the four primary conic sections produced?

circle, ellipse, parabola, hyperbola

b) A plane (through the vertex) intersects a double-napped cone perpendicular to the axis of the cone. This plane is slowly rotated through an angle of 90° forming different conics as it rotates. In which order are the degenerate conic sections produced?

point, single line, two intersecting lines

Multiple Choice

10. Which of these is a limiting case of an ellipse?

A. two parallel lines **B.** two intersecting lines
C. a circle **D.** a single line

11. Which of the following is not a degenerate conic?

A. two parallel lines **B.** two intersecting lines
C. a cylinder **D.** a single line

12. A parabola is produced by cutting a cone parallel to the generator. What happens to the parabola as the cutting plane moves closer to the vertex?

A. it becomes wider **B.** it becomes narrower
C. it is unchanged **D.** it becomes an ellipse

Questions 13-16 are based on the following information

> A double-napped cone is formed by rotating a line (the generator) about a vertical axis.
>
> The angle between the axis and the generator is 20°.
>
> A plane intersects the double-napped cone at an angle of θ.

13. In order for the conic section produced to be a hyperbola which must be true?

A. $0° \leq \theta < 20°$ **B.** $\theta = 20°$
C. $20° < \theta < 90°$ **D.** $\theta = 90°$

14. In order for the conic section produced to be a parabola which must be true?

A. $0° \leq \theta < 20°$ **B.** $\theta = 20°$
C. $20° < \theta < 90°$ **D.** $\theta = 90°$

15. In order for the conic section produced to be a circle which must be true?

A. $0° \leq \theta < 20°$ **B.** $\theta = 20°$
C. $20° < \theta < 90°$ **D.** $\theta = 90°$

16. In order for the conic section produced to be an ellipse which must be true?

A. $0° \leq \theta < 20°$ **B.** $\theta = 20°$
C. $20° < \theta < 90°$ **D.** $\theta = 90°$

17. The degenerate of a hyperbola is

A. two parallel lines **B.** two intersecting lines
C. a point **D.** a single line

18. A degenerate of a circle or ellipse is

A. two parallel lines **B.** two intersecting lines
C. a point **D.** a single line

19. A degenerate of a parabola is

A. a circle **B.** two intersecting lines
C. a point **D.** a single line

20. If the vertex of a double-napped cone is extended infinitely, the limiting position of the cone is

A. a circle **B.** a line
C. a point **D.** a cylinder

Answer Key

1. **a)** primary : ellipse, circle, parabola, hyperbola
b) degenerate : point, single line, two parallel lines, two intersecting lines, no locus

2. **a)** circle **b)** hyperbola **c)** parabola **d)** ellipse or circle **e)** point
f) line **g)** hyperbola **h)** two intersecting lines **i)** circle

3. **a)** perpendicular to the axis and not through the vertex
b) through one nappe at an angle greater than the generator angle and less than 90° and not through the vertex
c) parallel to the generator and not through the vertex
d) through both nappes at an angle between 0° and the generator angle and not through the vertex
e) parallel to the generator and through the vertex
f) through both nappes at an angle between 0° and the generator angle and through the vertex
g) through the vertex at an angle greater than the generator angle and less than or equal to 90°

4. two parallel lines **5.** hyperbola, two intersecting lines

6. circle, ellipse, single line, two parallel lines, no locus **7.** parabola

8. **a)** ellipse **b)** adjust the angle to 90°

9. **a)** circle, ellipse, parabola, hyperbola **b)** point, single line, two intersecting lines

10. C **11.** C **12.** B **13.** A **14.** B **15.** D

16. C **17.** B **18.** C **19.** D **20.** D

Conic Sections Lesson #2:
The Equation of a Conic Section in General Form

The General Form of the Equation of a Conic Section

The general form for the equation of a straight line is $y = mx + b$.
The general form for the equation of a quadratic function is $y = ax^2 + bx + c$.

The general form for the equation of a conic section (also called a **quadratic relation**) is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A, B, C, D, E, F \in \Re$.

The letters A, B, C, D, E, and F are called ***parameters***. The type of conic depends on the values of the parameters.

In this unit, we will only consider conics where $B = 0$ and where $A, B, C, D, E, F \in I$

General Form for a Quadratic Relation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Warm-Up #1 — *Observations of the Parameters A and C*

This warm-up will require the use of a computer with a graphing program (eg. "Zap-A-Graph") or a graphing calculator with a conics program.

Part 1 $A = C$

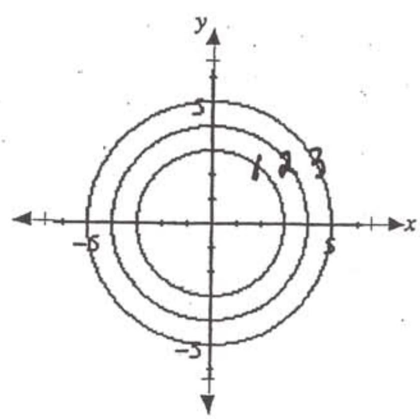
a) Write the specified parameters in each case and sketch each graph.

Equation 1	Equation 2	Equation 3
$x^2 + y^2 - 9 = 0$	$x^2 + y^2 - 16 = 0$	$2x^2 + 2y^2 - 50 = 0$
$A =$ 1	$A =$ 1	$A =$ 2
$B =$ 0	$B =$ 0	$B =$ 0
$C =$ 1	$C =$ 1	$C =$ 2
$D =$ 0	$D =$ 0	$D =$ 0
$E =$ 0	$E =$ 0	$E =$ 0
$F =$ −9	$F =$ −16	$F =$ −50

b) Complete the following statement:

In the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$,

the conic produced when $A = C$ is a circle.

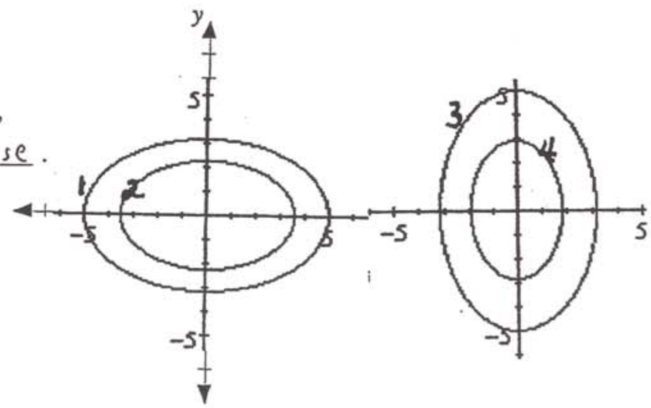


Part 2 *A and C both have the same sign, with $A \neq C$,* (*i.e.* $AC > 0$)

a) Write the specified parameters in each case and sketch each graph.

Equation 1	Equation 2	Equation 3	Equation 4
$2x^2 + 5y^2 - 50 = 0$	$-2x^2 - 5y^2 + 25 = 0$	$5x^2 + 2y^2 - 50 = 0$	$-7x^2 - 3y^2 + 25 = 0$
$A =$ 2	$A =$ -2	$A =$ 5	$A =$ -7
$B =$ 0	$B =$ 0	$B =$ 0	$B =$ 0
$C =$ 5	$C =$ -5	$C =$ 2	$C =$ -3
$D =$ 0	$D =$ 0	$D =$ 0	$D =$ 0
$E =$ 0	$E =$ 0	$E =$ 0	$E =$ 0
$F =$ -50	$F =$ 25	$F =$ -50	$F =$ 25

b) Complete the following statements:



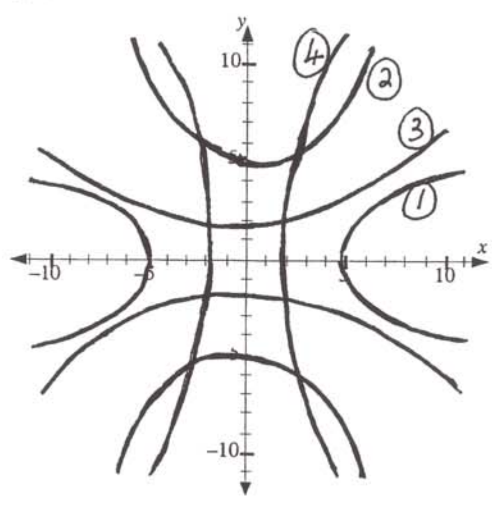
- In the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, the conic produced when $AC > 0$ is an ellipse.
- When $|A| < |C|$ the conic is a **horizontal** ellipse and has a shape like
- When $|A| > |C|$ the conic is a **vertical** ellipse and has a shape like 0

Part 3 *A and C have opposite signs,* (*i.e.* $AC < 0$)

a) Write the specified parameters in each case and sketch each graph.

Equation 1	Equation 2	Equation 3	Equation 4
$2x^2 - 5y^2 - 50 = 0$	$-5x^2 + 2y^2 - 50 = 0$	$3x^2 - 7y^2 + 25 = 0$	$-7x^2 + 3y^2 + 25 = 0$
$A =$ 2	$A =$ -5	$A =$ 3	$A =$ -7
$B =$ 0	$B =$ 0	$B =$ 0	$B =$ 0
$C =$ -5	$C =$ 2	$C =$ -7	$C =$ 3
$D =$ 0	$D =$ 0	$D =$ 0	$D =$ 0
$E =$ 0	$E =$ 0	$E =$ 0	$E =$ 0
$F =$ -50	$F =$ -50	$F =$ 25	$F =$ 25

b) Complete the following statements:



- In the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, the conic produced when $AC < 0$ is a hyperbola.
- Equations 1 and 4 open along the horizontal axis.
- Equations 2 and 3 open along the vertical axis.
- If the values of A and C are interchanged, and F is not changed, then the direction of opening changes.

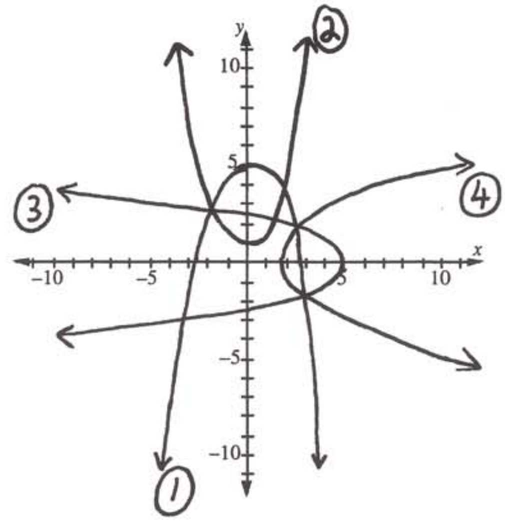
Part 4 $A = 0$, or $C = 0$, *but NOT BOTH* (*i.e.* $AC = 0$)

a) Write the specified parameters in each case and sketch each graph

Equation 1	Equation 2	Equation 3	Equation 4
$x^2 + y - 5 = 0$	$5x^2 - y + 1 = 0$	$y^2 + x - 5 = 0$	$-2y^2 + 3x - 5 = 0$
$A = 1$	$A = 5$	$A = 0$	$A = 0$
$B = 0$	$B = 0$	$B = 0$	$B = 0$
$C = 0$	$C = 0$	$C = 1$	$C = -2$
$D = 0$	$D = 0$	$D = 1$	$D = 3$
$E = 1$	$E = -1$	$E = 0$	$E = 0$
$F = -5$	$F = 1$	$F = -5$	$F = -5$

b) Complete the following statements:

- In the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, the conic produced when $AC = 0$ is a parabola.
- When $A = 0$ and $C \neq 0$, the conic opens left or right and has a shape like $\supset$ or $\subset$
- When $A \neq 0$ and $C = 0$, the conic opens up or down and has a shape like $\cup$ or $\cap$



General Effects of the Parameter A and C

Circle

If $A = C$, then the conic is a circle.

Ellipse

If $A \neq C$ **<u>and</u>** they have the same sign (i.e. $AC > 0$), the conic is an ellipse.

If $|A| > |C|$ then it takes the shape 

If $|A| < |C|$ then it takes the shape 

The closer in value A is to C, the closer an ellipse is to a circle.

Hyperbola

If A and C have different signs (i.e. $AC < 0$), then the conic is a hyperbola.

If A and C are interchanged, and F remains constant,
the direction of opening will change.

The hyperbola has asymptotes which will be discussed in a later lesson.

Parabola

If A or C, **<u>not both</u>**, equal zero, then the conic is a parabola.

If $A \neq 0$ and $C = 0$,
then the parabola opens up,  (eg. $y = x^2$) or down,  (eg. $y = -x^2$)

If $A = 0$ and $C \neq 0$,
then the parabola opens right,  (eg. $x = y^2$) or left,  (eg. $x = -y^2$)

Although NOT part of the curriculum, the following two points may be of interest:

- All the conics in parts 1 to 3 of the Warm-Up have their "centre" located at the origin. This is because the parameters D and E are both zero. In general changing the parameters D and E will affect the location of the conic, left or right (D), or up or down (E), from the origin.
- Changing the parameter F has a wide variety of effects, but generally involves a change in size of the conic, which may result in a degenerate conic.

State the type of conic and sketch the approximate shape of the conic represented by each of the following equations.

a) $x^2 + 3y^2 - 6x + 8y - 90 = 0$ $\quad AC > 0 \quad A < C \quad$ ellipse 

b) $3x^2 + 3y^2 - 4x + 5y - 63 = 0$ $\quad A = C \quad$ circle 

c) $x^2 - 4x + y - 20 = 0$ $\quad AC = 0 \quad A \neq 0,\ C = 0 \quad$ parabola 

A quadratic relation has equation $x^2 + 2y^2 - 6x - 4y - 16 = 0$.

a) Which type of conic is represented by the equation? ellipse

b) Determine the x and y-intercepts. Use the intercepts to make a sketch of the conic.

x-intercepts

$y = 0$

$x^2 - 6x - 16 = 0$

$(x + 2)(x - 8) = 0$

x intercepts $-2, 8$

y-intercepts

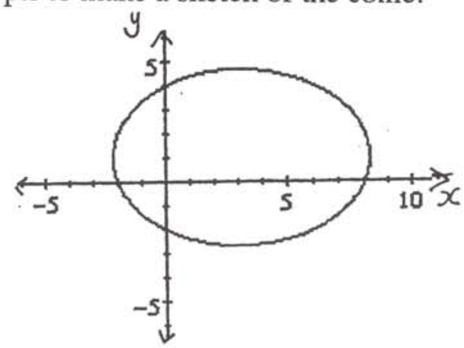
$x = 0$

$2y^2 - 4y - 16 = 0$

$2(y^2 - 2y - 8) = 0$

$2(y + 2)(y - 4) = 0$

y intercepts $-2, 4$



c) The values of A and C are interchanged.

i) How would the shape of the original conics be changed?

The ellipse would change from a horizontal ellipse to a vertical ellipse.

ii) Determine the x and y-intercepts of this new conic and make a sketch.

$2x^2 + y^2 - 6x - 4y - 16 = 0$

x-intercepts

$y = 0$

$2x^2 - 6x - 16 = 0$

$2(x^2 - 3x - 8) = 0$

use the quadratic formula

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{3 \pm \sqrt{9 + 32}}{2}$

$x_{int.} = \dfrac{3 \pm \sqrt{41}}{2}$ $(-1.70, 4.70)$

y-intercepts

$x = 0$

$y^2 - 4y - 16 = 0$

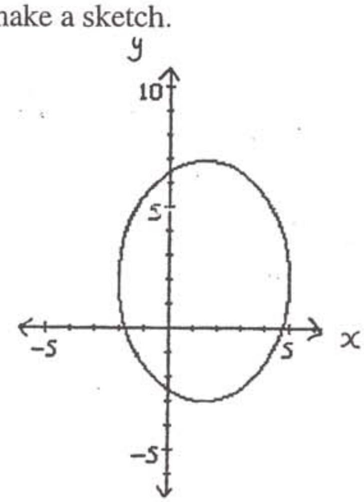
quad form.

$y = \dfrac{4 \pm \sqrt{16 + 64}}{2}$

$= \dfrac{4 \pm \sqrt{80}}{2}$

$y_{int.} = 2 \pm 2\sqrt{5}$

$(-2.47, 6.47)$



d) What value of A would make the original conic into a circle?

$A = 2$

Complete Assignment Questions #1 - #18

Assignment

1. Which conic is represented by each equation?

a) $3x^2 + 3y^2 + 12x + 4y - 54 = 0$

circle $A = C$

b) $3x^2 - 2y^2 + 7x - 14y - 57 = 0$

hyperbola $AC < 0$

c) $x^2 + 2y^2 - 7x + 4y - 21 = 0$

ellipse $AC > 0$

d) $x^2 + 5x - 2y - 7 = 0$

parabola $C = 0$

e) $x^2 - y^2 - x - 8y - 50 = 0$

hyperbola $AC < 0$

f) $-3x^2 + 3y^2 - 4x + 5y - 63 = 0$

hyperbola $AC < 0$

2. State the type of conic and sketch the approximate shape of the conic represented by each of the following equations.

a) $2x^2 + 2y^2 - 4x + 8y - 40 = 0$

circle

b) $7x^2 + 3y^2 - 3x + 5y - 80 = 0$

ellipse $|A| > |C|$

c) $-4x + y^2 - 20 = 0$ $y^2 = 4x + 20$

parabola

of the form $x = y^2$

d) $x^2 + 3x + 5y - 21 = 0$ $5y = -x^2 - 3x + 21$

parabola

of the form $y = -x^2$

e) $-2x^2 - 6y^2 + 8x - y + 75 = 0$

ellipse $|A| < |C|$

f) $x^2 + 3x - 5y - 21 = 0$ $x^2 + 3x - 21 = 5y$

parabola

of the form $y = x^2$

3. Answer the following questions based on the equation $6x^2 + 2y^2 - 9x + 14y - 68 = 0$.

a) Which conic is represented by the equation? ellipse

b) What value of A would transform the conic into a circle? 2

c) What value of C would transform the original conic into a circle? 6

d) What change would take place if the values of A and C were interchanged?

the ellipse would change from a vertical ellipse to a horizontal ellipse .

4. Consider the equation $x^2 + y^2 - 6x + 10y + 9 = 0$

a) Which quadratic relation is represented by the equation?

circle

b) Susan used a computer program to graph the quadratic relation. The resulting graph was a hyperbola. When she checked the equation on her computer, she found she had entered one of the signs incorrectly. Which of the four signs was incorrectly entered?

the "+" in front of the y^2 should be "−".

c) Determine the *x* and *y*-intercepts of the original equation and sketch the graph.

x-int. $y=0$

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

$x = 3$

x-intercept: 3

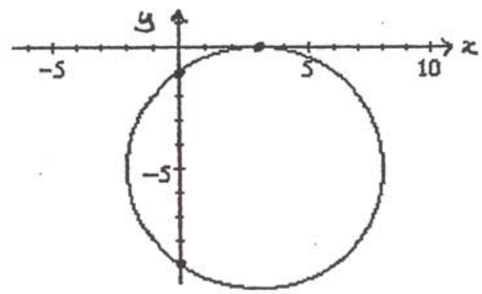
y-int. $x=0$

$y^2 + 10y + 9 = 0$

$(y+9)(y+1) = 0$

$y = -9, -1$

y-intercepts: $-9, -1$



5. Answer the following questions given the equation $3x^2 - y^2 - 36x + y + 60 = 0$

a) What type of curve does this equation represent? hyperbola

b) Determine the *x* and *y*-intercepts of the original equation and sketch the graph.

x-int. $y=0$

$3x^2 - 36x + 60 = 0$

$3(x^2 - 12x + 20) = 0$

$3(x-2)(x-10) = 0$

$x = 2, 10$

x-intercepts: 2, 10

y-int. $x=0$

$-y^2 + y + 60 = 0$

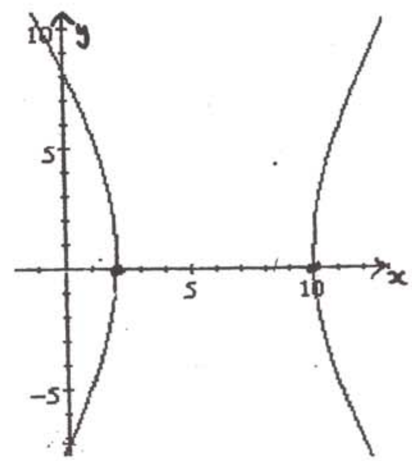
$y^2 - y - 60 = 0$

quadratic formula

$$y = \frac{1 \pm \sqrt{1+240}}{2}$$

$$y = \frac{1 \pm \sqrt{241}}{2}$$

y-intercepts: 8.26, −7.26



c) If the values of *A* and *C* are interchanged in the equation, what effect will this have on the basic shape of the graph.

the hyperbola would open along a vertical axis i.e

rather than along a horizontal axis.

Multiple Choice

6. Which equation represents an ellipse?

A. $-2x^2 - 2y^2 - 100 = 0$ no values possible
B. $-2x^2 + 2y^2 + 100 = 0$ $AC < 0$
C. $x^2 - 2y - 100 = 0$ $C = 0$
D. $-x^2 - 2y^2 + 100 = 0$ $AC > 0$

7. Which equation represents a hyperbola?

A. $-2x^2 + 2y^2 - 100 = 0$ $AC < 0$
B. $2x^2 + 2y^2 + 100 = 0$
C. $-x^2 - 2y^2 - 100 = 0$
D. $-x^2 - 2y^2 + 100 = 0$

8. Which equation represents a non-degenerate parabola?

A. $2y^2 - 3x + 10 = 0$
B. $2x^2 - 3x + 10 = 0$ x^2 needs a 'y' value
C. $x^2 - 2y^2 - 3x - 10 = 0$ $AC < 0$
D. $x^2 - 2y^2 - 3y + 10 = 0$ $AC < 0$

9. In the equation
$Ax^2 + Cy^2 + 8x + 10y - 34 = 0$
and either A or $C = 0$, then the curve is

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

10. In the equation
$Ax^2 + Bxy + Cy^2 + 8x + 10y - 34 = 0$
if $B = 0$ and $A, C > 0, A = C,$
then the curve is

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

11. In the equation
$Ax^2 + Cy^2 + 6x - 10y + 40 = 0,$
$AC < 0, A \neq C,$ then the curve is

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

12. In the equation
$Ax^2 + Cy^2 + 6x - 10y + 40 = 0,$ $A < 0$
$A, C < 0, A \neq C,$ then the curve is $C < 0$
$AC > 0$

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

13. In the equation
$Ax^2 + Bxy + Cy^2 + 8x + 10y - 34 = 0$
if $A = B = C = 0,$ then the curve is the degenerate of

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

$8x + 10y - 34 = 0$ is the equation of a line.

14. The equation
$Ax^2 + Cy^2 + Dx + Ey + F = 0$
represents a hyperbola if

A. $AC > 0, A \neq C$
B. $AC < 0, A \neq C$
C. $AC = 0$
D. $A = C$

15. The conic given by
$Ax^2 - 2y^2 + 14 = 0$ with
$A < 0$ and $A \neq -2$ is $AC > 0$

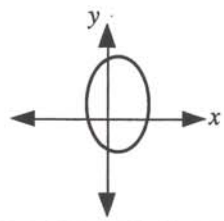
A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

16. In the equation $Ax^2 + Cy^2 + F = 0$ if $B = 0$ and $A, C > 0, A < C$, then the curve is

A. an ellipse with the longer axis along the x-axis.
B. an ellipse with the longer axis along the y-axis.
C. a hyperbola along the x-axis.
D. a hyperbola along the y-axis.

Use the following information to answer the next question.

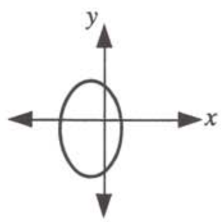
The graph of the quadratic relation $2x^2 + y^2 - 3x - 2y - 25 = 0$ is shown.



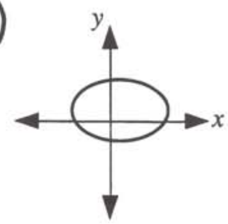
17. The resulting graph when the coefficients A and C are interchanged is

ellipse with $|A| < |C|$

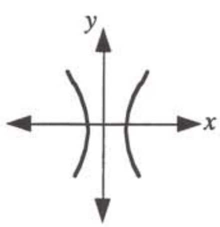
A.



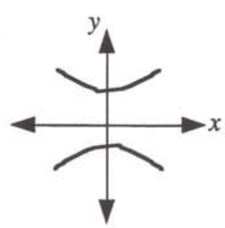
B.



C.



D.



Numerical Response **18.** The positive y-intercept, to the nearest hundredth, of the parabola $y^2 - 2x + 3y - 7 = 0$ is _____ .
(Record your answer in the numerical response box from left to right)

1	.	5	4

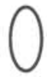
$x = 0 \qquad y^2 + 3y - 7 = 0$

quadratic formula

$$y = \frac{-3 \pm \sqrt{9 + 28}}{2} = -4.54,\ 1.54$$

Answer Key

1. **a)** circle **b)** hyperbola **c)** ellipse **d)** parabola **e)** hyperbola **f)** hyperbola

2. **a)** circle  **b)** ellipse  **c)** parabola 

d) parabola  **e)** ellipse  **f)** parabola 

3. **a)** ellipse **b)** 2 **c)** 6
d) ellipse would have its longer axis parallel to the x-axis (i.e. horizontal ellipse).

4. **a)** circle **b)** the + sign in front of the y^2 was entered as a − sign.
c) x-intercept = 3, y-intercepts = -9 and -1

5. **a)** hyperbola **b)** x-intercepts = 2 and 10, y-intercepts = $\dfrac{1 \pm \sqrt{241}}{2} \approx -7.26$ and 8.26
c) hyperbola would open along the vertical axis

6. D **7.** A **8.** A **9.** B **10.** A

11. D **12.** C **13.** B **14.** B **15.** C

16. A **17.** B **18.**

1	.	5	4

Conic Sections Lesson #3:
The Equation of a Conic Section in Standard Form

Warm-Up #1

The four equations below represent the equations of different conic sections, but they are in a format we are not yet familiar with.

i) $x-3=2(y+3)^2$ **ii)** $(x-2)^2+(y+1)^2=4$

iii) $\dfrac{x^2}{9}+\dfrac{(y+1)^2}{25}=1$ **iv)** $\dfrac{(x-5)^2}{4}-\dfrac{(y-2)^2}{16}=1$

- Convert each equation into the general form $Ax^2+Cy^2+Dx+Ey+F=0$ and state which type of conic section each equation represents.

i) $x-3=2(y^2+6y+9)$

$x-3=2y^2+12y+18$

$\underline{\underline{2y^2-x+12y+21=0}}$

parabola

ii) $x^2-4x+4+y^2+2y+1=4$

$\underline{\underline{x^2+y^2-4x+2y+1=0}}$

circle

iii) $225\left(\dfrac{x^2}{9}\right)+\dfrac{225(y+1)^2}{25}=225(1)$

$25x^2+9(y+1)^2=225$

$25x^2+9(y^2+2y+1)=225$

$25x^2+9y^2+18y+9=225$

$\underline{\underline{25x^2+9y^2+18y-216=0}}$

ellipse

iv) $16\left[\dfrac{(x-5)^2}{4}\right]-16\left[\dfrac{(y-2)^2}{16}\right]=16(1)$

$4(x-5)^2-1(y-2)^2=16$

$4(x^2-10x+25)-(y^2-4y+4)=16$

$4x^2-40x+100-y^2+4y-4=16$

$\underline{\underline{4x^2-y^2-40x+4y+80=0}}$

hyperbola

Warm-Up #2

The equations in Warm-Up #1 are examples of conic sections whose equations are written in **standard form**.

Use a graphing program such as *Zap-a-Graph* to sketch the conics from Warm-Up #1. Use general form and standard form to verify the graphs are identical.

The Standard Form of the Equation of a Conic Section

The equations listed below are the **standard form** of the equation of each type of conic section.

Parabola

- Opening up or down

$$y - k = a(x - h)^2$$

- Opening left or right

$$x - h = a(y - k)^2$$

Circle

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{or} \qquad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1, \qquad \text{where } a = r$$

Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

- For a horizontal ellipse, $a^2 > b^2$.
- For a vertical ellipse, $a^2 < b^2$.

Hyperbola

- Opening along the x-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

- Opening along the y-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$$

or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

- The hyperbola has asymptotes with slopes equal to $\pm \dfrac{b}{a}$.

The above information is **NOT** all on the formula sheet. See the formula below for the standard form of conics equations

Standard Form for a Quadratic Relation

$$\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = \pm 1$$

$$y - k = a(x - h)^2$$

$$x - h = a(y - k)^2$$

These formulas are on the formula sheet

In the next lesson we will use the standard form to identify features of the graph of a conic section such as centre, intercepts, domain, and range.

Class Ex. #1

Identify the type of conic section from the equation. Do not use technology.

a) $x^2 + y^2 = 16$ circle

b) $x + 5 = \frac{1}{2}(y - 3)^2$ parabola

c) $\frac{(x-4)^2}{4} - y^2 = 1$ hyperbola

Complete Assignment Questions #1 - #2

Assignment

1. State the type of conic section which each equation represents.

a) $(x-2)^2 + (y-4)^2 = 64$ circle

b) $y - 2 = 5x^2$ parabola

c) $\frac{(x-7)^2}{8} + \frac{(y+2)^2}{20} = 1$ ellipse

d) $(x-2)^2 - (y-4)^2 = 64$ hyperbola

e) $x - 2 = 4(y-3)^2$ parabola

f) $\frac{(y-7)^2}{8} - \frac{(x+2)^2}{20} = 1$ hyperbola

2. For each of the following quadratic relations defined in standard form;

i) State the type of conic section represented

ii) Convert the equation into general form.

a) $\frac{(x-3)^2}{4} + \frac{(y+2)^2}{4} = 1$ circle

$$4\left[\frac{(x-3)^2}{4}\right] + 4\left[\frac{(y-2)^2}{4}\right] = 4(1)$$

$$(x-3)^2 + (y-2)^2 = 4$$

$$x^2 - 6x + 9 + y^2 - 4y + 4 = 4$$

$$\underline{\underline{x^2 + y^2 - 6x + 4y + 9 = 0}}$$

b) $\frac{x^2}{4} - \frac{(y+5)^2}{9} = 1$ hyperbola

$$36\left(\frac{x^2}{4}\right) - 36\left[\frac{(y+5)^2}{9}\right] = 36(1)$$

$$9x^2 - 4(y+5)^2 = 36$$

$$9x^2 - 4(y^2 + 10y + 25) = 36$$

$$9x^2 - 4y^2 - 40y - 100 = 36$$

$$\underline{\underline{9x^2 - 4y^2 - 40y - 136 = 0}}$$

c) $y + 1 = 4(x - 6)^2$ parabola

$y + 1 = 4(x^2 - 12x + 36)$

$y + 1 = 4x^2 - 48x + 144$

$\underline{\underline{4x^2 - 48x - y + 143 = 0}}$

d) $\dfrac{(x+4)^2}{9} + \dfrac{(y-4)^2}{36} = 1$ ellipse

$36\left[\dfrac{(x+4)^2}{9}\right] + 36\left[\dfrac{(y-4)^2}{36}\right] = 36(1)$

$4(x+4)^2 + (y-4)^2 = 36$

$4(x^2 + 8x + 16) + y^2 - 8y + 16 = 36$

$4x^2 + 32x + 64 + y^2 - 8y + 16 = 36$

$\underline{\underline{4x^2 + y^2 + 32x - 8y + 44 = 0}}$

Answer Key

1. a) circle **b)** parabola **c)** ellipse **d)** hyperbola **e)** parabola **f)** hyperbola

2. a) i) circle **ii)** $x^2 + y^2 - 6x + 4y + 9 = 0$ **b) i)** hyperbola **ii)** $9x^2 - 4y^2 - 40y - 136 = 0$

c) i) parabola **ii)** $4x^2 - 48x - y + 143 = 0$ **d) i)** ellipse **ii)** $4x^2 + y^2 + 32x - 8y + 44 = 0$

Conic Sections Lesson #4:
Transformations of Parabolas

Review of Transformations

Recall the following transformations which are relevant to this unit.

Replacement for x or y	*Translation*
$x \rightarrow \frac{1}{a}x$	horizontal stretch by a factor of a about the y-axis
$y \rightarrow \frac{1}{b}y$	vertical stretch by a factor of b about the x-axis
$x \rightarrow x-h$	horizontal translation h units right
$y \rightarrow y-k$	vertical translation k units up
$x \rightarrow -x$	reflection in the y-axis
$y \rightarrow -y$	reflection in the x-axis

Describe how the graph of the second relation compares to the graph of the first relation.

a) $y = x^2$

$y - 3 = x^2$

$y \rightarrow y - 3$

vertical translation 3 units up

b) $x^2 + y^2 = 1$

$(x-2)^2 + (y+3)^2 = 1$

$x \rightarrow x - 2$

$y \rightarrow y + 3$

translation 2 units right and 3 units down

c) $x^2 + y^2 = 1$

$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{5}y\right)^2 = 1$

$x \rightarrow \frac{1}{2}x$

$y \rightarrow \frac{1}{5}y$

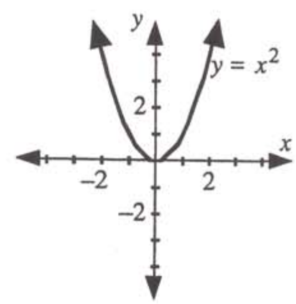
horizontal stretch by a factor of 2 about the y-axis and a vertical by a factor of 5 about the x-axis

Transformations of the Parabola $y = x^2$

The graph of the parabola with equation $y = x^2$ is shown.

Complete the following for the graph.

- domain $x \in R$
- range $y \geq 0$
- coordinates of vertex $(0, 0)$



Part 1 *Transforming the Parabola using a Numerical Value for the Stretch Factor*

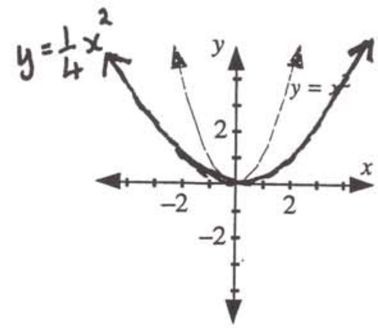
x is replaced by $\frac{1}{2}x$ to get the equation $y = \left(\frac{1}{2}x\right)^2$ or $y = \frac{1}{4}x^2$.

a) Complete the following for the graph of the equation $y = \frac{1}{4}x^2$.

- The transformation from $y = x^2$ is a horizontal stretch by a factor of 2 about the y-axis.

b) Draw the transformed image on the grid and complete.

- domain $x \in R$
- range $y \geq 0$
- coordinates of vertex $(0, 0)$



Part 2A *Transforming the Parabola using an Algebraic Value for the Stretch Factor*

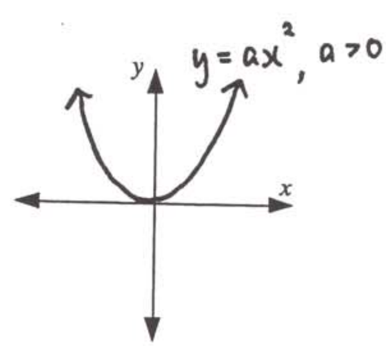
x is replaced by $\sqrt{a}\,x$ (where $a > 0$) to get the equation $y = \left(\sqrt{a}\,x\right)^2$ or $y = ax^2$.

a) Complete the following for the graph of the equation $y = ax^2$ where $a > 0$.

- The transformation from $y = x^2$ is a horizontal stretch by a factor of $\frac{1}{\sqrt{a}}$ about the y-axis

b) Draw the transformed image on the grid and complete.

- domain $x \in R$
- range $y \geq 0$
- coordinates of vertex $(0, 0)$



Part 2B ***Transforming the Parabola with a Stretch and Translations***

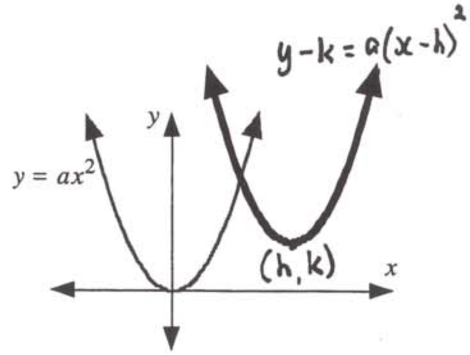
- x is then replaced by $x - h$ and y is then replaced by $y - k$ to get the equation
  $y - k = a(x - h)^2$, where $a > 0$.

a) Complete for the graph of the equation $y - k = a(x - h)^2$.

- The transformation from $y = ax^2$ is a translation h units right and k units up.

b) Label the transformed image on the grid and complete.

- domain $x \in R$
- range $y \geq k$
- coordinates of vertex (h, k)



c) What changes, if any, would there be to the answers to b) if the equation was in the form $y - k = a(x - h)^2$, where $a < 0$?

If $a < 0$ there is a reflection in the x-axis before the translation so the range would be $y \leq k$

Features of the Graph of the Parabola $y - k = a(x - h)^2$

The parabola defined by the equation $y - k = a(x - h)^2$ has

- vertex (h, k)
- domain $x \in \Re$.
- If $a > 0$ the range is $y \geq k$ and if $a < 0$ the range is $y \leq k$.
- x- and y-intercepts are determined by solving the equations $y = 0$ and $x = 0$ respectively.

Compared to the graph of $y = x^2$ the graph of $y = ax^2$, $a > 0$, can be regarded as either

- **a horizontal stretch by a factor of $\frac{1}{\sqrt{a}}$ about the y-axis or**
- **a vertical stretch by a factor of a about the x-axis.**

In this lesson, we will use the horizontal stretch as it will help us with transformations of circles, ellipses, and hyperbolas.

Consider the conic section with equation $y - 9 = -9(x - 2)^2$. $\quad y - k = a(x - h)^2$

a) Describe the series of transformations which would transform the graph $y = x^2$ to the graph of $y - 9 = -9(x - 2)^2$.

$x \to 3x \quad y = (3x)^2 \quad y = 9x^2$

$y \to -y \quad -y = 9x^2 \quad y = -9x^2$

$\left.\begin{matrix} x \to x-2 \\ y \to y-9 \end{matrix}\right\} \quad y - 9 = -9(x-2)^2$

horizontal stretch by a factor of $\frac{1}{3}$ about the y-axis, a reflection in the x-axis then a translation 2 units right and 9 units up.

b) Determine the following features of the graph of $y - 9 = -9(x - 2)^2$.

i) domain $\quad x \in R$

ii) range $\quad y \leq 9$

iii) coordinates of the vertex

$(2, 9)$

iv) x- and y-intercepts

x-int. $y = 0$

$-9 = -9(x-2)^2$

$1 = (x-2)^2$

$x - 2 = \pm 1$

$x = 1, 3$

x-intercepts are 1, 3

y-int $x = 0$

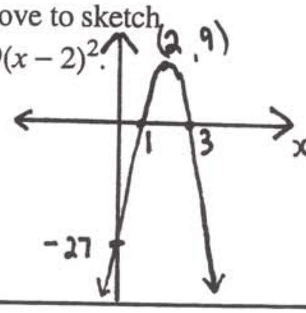
$y - 9 = -9(4)$

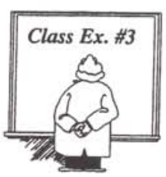
$y - 9 = -36$

$y = -27$

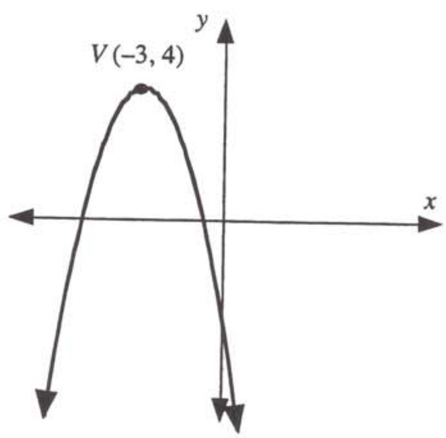
y-intercept is -27

c) Use the information above to sketch the graph of $y - 9 = -9(x - 2)^2$.





The graph shown has equation $y - k = -(x - h)^2$.
The vertex has coordinates $V(-3, 4)$. $\quad V(h, k)$



a) Determine the values of h and k.

$h = -3, \; k = 4$

b) Write the equation in general form.

$y - 4 = -(x+3)^2$

$y - 4 = -(x^2 + 6x + 9)$

$y - 4 = -x^2 - 6x - 9$

$\underline{\underline{x^2 + 6x + y + 5 = 0}}$

c) Find the x- and y-intercepts.

x-int $y = 0$

$x^2 + 6x + 5 = 0$

$(x+5)(x+1) = 0$

x-intercepts are $-5, -1$

y-int $x = 0$

$y + 5 = 0$

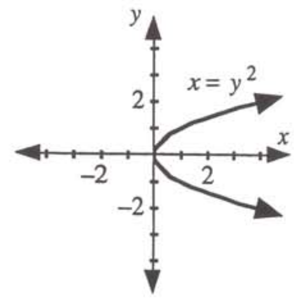
y intercept $= -5$

Complete Assignment Questions #1 - #3

Transformations of the Parabola $x = y^2$

The graph of the parabola with equation $x = y^2$ is shown.

Complete the following for the graph.



- domain $x \geq 0$
- range $y \in R$
- coordinates of vertex $(0, 0)$

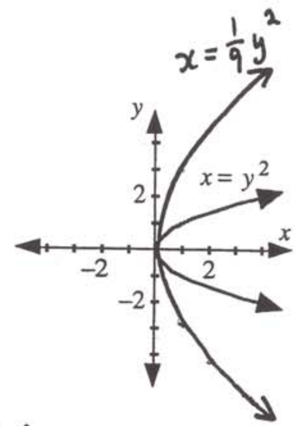
Part 1 *Transforming the Parabola using a Numerical Value for the Stretch Factor*

y is replaced by $\frac{1}{3}y$ to get the equation $x = \left(\frac{1}{3}y\right)^2$ or $x = \frac{1}{9}y^2$.

a) Complete the following for the graph of the equation $x = \frac{1}{9}y^2$.

- The transformation from $x = y^2$ is a vertical stretch by a factor of 3 about the x-axis.

b) Draw the transformed image on the grid and complete.



- domain $x \geq 0$
- range $y \in R$
- coordinates of vertex $(0, 0)$

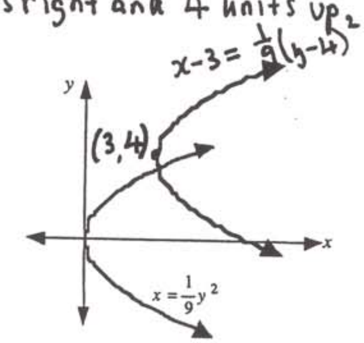
Part 2 *Transforming the Parabola with a Stretch and Translations*

x is then replaced by $x - 3$ and y by $y - 4$ to get the equation $x - 3 = \frac{1}{9}(y - 4)^2$.

a) Complete for the graph of the equation $x - 3 = \frac{1}{9}(y - 4)^2$.

- The transformation from $x = \frac{1}{9}y^2$ is a translation 3 units right and 4 units up

b) Label the transformed image on the grid and complete.



- domain $x \geq 3$
- range $y \in R$
- coordinates of vertex $(3, 4)$

c) What changes, if any, would there be to the answers to b) if the equation was $x - 3 = \frac{1}{9}(y - 4)^2$?

Features of the Graph of the Parabola $x - h = a(y - k)^2$

The parabola defined by the equation $x - h = a(y - k)^2$ has

- vertex (h, k)
- range $y \in \Re$.
- If $a > 0$ the domain is $x \geq h$ and if $a < 0$ the domain is $x \leq h$.
- x- and y-intercepts are determined by solving the equations $y = 0$ and $x = 0$ respectively.

Note

Compared to the graph of $x = y^2$, the graph of $x = ay^2$, $a > 0$, can be regarded as either

- **a vertical stretch by a factor of $\frac{1}{\sqrt{a}}$ about the x-axis or**
- **a horizontal stretch by a factor of a about the y-axis.**

In this lesson, we will use the vertical stretch as it will help us with transformations of circles, ellipses, and hyperbolas.

Class Ex. #4

Consider the conic section with equation $x - 3 = \frac{1}{16}(y + 4)^2$. $\quad x - h = a(y-k)^2$

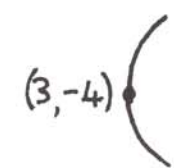
a) Describe the series of transformations which would transform the graph $x = y^2$ to the graph of $x - 3 = \frac{1}{16}(y + 4)^2$.

vertical stretch by a factor of 4 about the x-axis then a translation 3 units right and 4 units down.

$y \to \frac{1}{4}y \quad x = \left(\frac{1}{4}y\right)^2 \quad x = \frac{1}{16}y^2$

$\left.\begin{array}{l} x \to x-3 \\ y \to y+4 \end{array}\right\} \quad x - 3 = \frac{1}{16}(y+4)^2$

b) Determine the following features of the graph of $x - 3 = \frac{1}{16}(y + 4)^2$.



i) domain

$x \geq 3$

ii) range

$y \in R$

iii) coordinates of the vertex

$(3, -4)$

iv) x- and y-intercepts

x-int $y = 0$

$x - 3 = \frac{1}{16}(4)^2$

$x - 3 = 1$

x intercept = 4

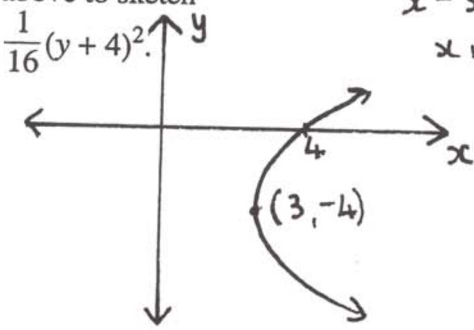
y-int. $x = 0$

$-3 = \frac{1}{16}(y+4)^2$

$-48 = (y+4)^2$

no y-intercept

c) Use the information above to sketch the graph of $x - 3 = \frac{1}{16}(y + 4)^2$.



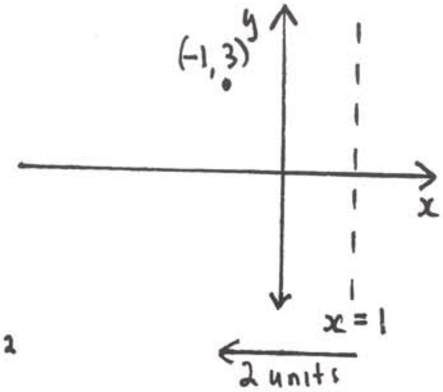
A graph has an equation of the form $x - h = -(y - k)^2$. The vertex has coordinates $(-1, 3)$.
h k

a) State the equation of the graph.

$x + 1 = -(y-3)^2$

b) Determine the equation of the parabola if it is stretched horizontally by a factor of 2 about the line $x = 1$.

the vertex is 2 units right of $x = 1$ so after the stretch the vertex will be $2(2) = 4$ units right of $x = 1$ at $(-3, 3)$. So $h = -3$ and $k = 3$.
From the equation $x - h = a(y-k)^2$, the horizontal stretch by a factor of 2 changes the value of a from -1 to -2.
The equation of the parabola is $\underline{\underline{x + 3 = -2(y-3)^2}}$



or the horizontal stretch by a factor of 2 about the line $x = 1$ is equivalent to a horizontal stretch by a factor of 2 about the y-axis in which the vertex becomes $(-2, 3)$ followed by a horizontal translation which translates the vertex $(-2, 3)$ to the final vertex at $(-3, 3)$ i.e 1 unit left

so $x \to \frac{1}{2}x$ $\qquad x + 1 = -(y-3)^2 \qquad \frac{1}{2}x + 1 = -(y-3)^2$

then $x \to x + 1 \qquad \frac{1}{2}(x+1) + 1 = -(y-3)^2 \qquad (x+1) + 2 = -2(y-3)^2$

$x + 3 = -2(y-3)^2$

or by formula:
$x \to \frac{1}{2}x$
then $x \to x + (F-1)L$
$x \to x + (2-1)(1)$
$x \to x + 1$

Complete Assignment Questions #4 - #9

Assignment

1. Determine the equation of the parabola $y = x^2$ after each of the following transformations:

a) translated 3 units down

$y \to y + 3 \qquad y + 3 = x^2$
or $y = x^2 - 3$

b) horizontal translation 5 units right

$x \to x - 5 \qquad y = (x-5)^2$

c) horizontal stretch by a factor of 4 about the line $x = 0$

$x \to \frac{1}{4}x \qquad y = \left(\frac{1}{4}x\right)^2$
$y = \frac{1}{16}x^2$

d) vertical stretch by a factor of $\frac{1}{9}$ about the line $y = 0$.

$y \to 9y \qquad 9y = x^2 \qquad y = \frac{1}{9}x^2$

2. Consider the conic section with equation $y+9=\frac{1}{4}(x-4)^2$.

a) Use transformations to describe how the graph of this conic section compares to the graph of the parabola $y=x^2$.

$x \to \frac{1}{2}x \quad y=\left(\frac{1}{2}x\right)^2 \quad y=\frac{1}{4}x^2$

$\left.\begin{array}{l} x \to x-4 \\ y \to y+9 \end{array}\right\} \quad y+9=\frac{1}{4}(x-4)^2$

horizontal stretch by a factor of 2 about the y-axis then a translation 4 units right and 9 units down.

b) Determine the following features of the graph of $y+9=\frac{1}{4}(x-4)^2$.

(4, -9)

i) domain $x \in R$

ii) range $y \geq 9$

iii) coordinates of the vertex $(4, -9)$

iv) x- and y-intercepts

x-int. $y=0$

$9=\frac{1}{4}(x-4)^2$

$36=(x-4)^2$

$\pm 6 = x-4$

x-intercepts: $-2, 10$

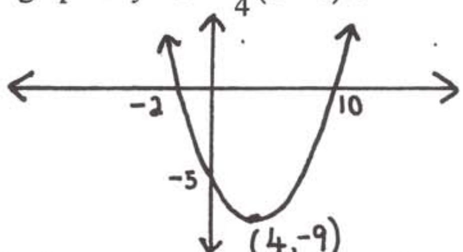
y-int. $x=0$

$y+9=\frac{1}{4}(-4)^2$

$y+9=4$

y-intercept: -5

c) Use the information above to sketch the graph of $y+9=\frac{1}{4}(x-4)^2$.



3. Consider the conic section with equation $y=-2(x+6)^2$.

a) Use transformations to describe how the graph of this conic section compares to the graph of the parabola $y=x^2$.

$x \to \sqrt{2}x \quad y=(\sqrt{2}x)^2 \quad y=2x^2$

$y \to -y \quad -y=2x^2 \quad y=-2x^2$

$x \to x+6 \quad y=-2(x+6)^2$

horizontal stretch by a factor of $\frac{1}{\sqrt{2}}$ about the y-axis, a reflection in the x-axis then a translation 6 units left.

b) Determine the following features of the graph of $y=-2(x+6)^2$.

(-6, 0)

i) domain $x \in R$

ii) range $y \leq 0$

iii) coordinates of the vertex $(-6, 0)$

iv) x- and y-intercepts

x-int. $y=0$

$0=-2(x+6)^2$

$x=-6$

x-intercept: -6

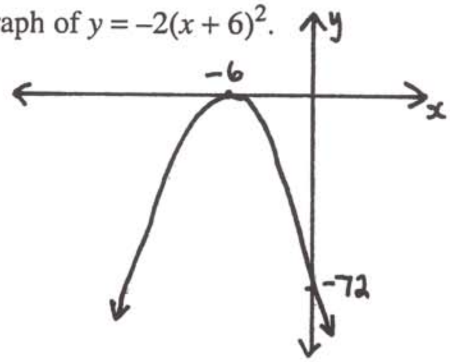
y-int. $x=0$

$y=-2(6^2)$

$y=-72$

y-intercept: -72

c) Use the information above to sketch the graph of $y=-2(x+6)^2$.



4. Consider the conic section with equation $x - 1 = -4(y - 2)^2$.

a) Use transformations to describe how the graph of this conic section compares to the graph of the parabola $x = y^2$.

$y \to 2y \quad x = (2y)^2 \quad x = 4y^2$

$x \to -x \quad -x = 4y^2 \quad x = -4y^2$

$\left.\begin{array}{l} x \to x-1 \\ y \to y-2 \end{array}\right\} \quad x - 1 = -4(y-2)^2$

vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, a reflection in the y-axis, then a translation 1 unit right and 2 units up.

b) Determine the following features of the graph of $x - 1 = -4(y - 2)^2$.

i) domain $x \leq 1$

ii) range $y \in R$

iii) coordinates of the vertex $(1, 2)$

iv) x- and y-intercepts

x-int. $y = 0$

$x - 1 = -4(-2)^2$

$x - 1 = -16$

$x = -15$

x-intercept: -15

y-int. $x = 0$

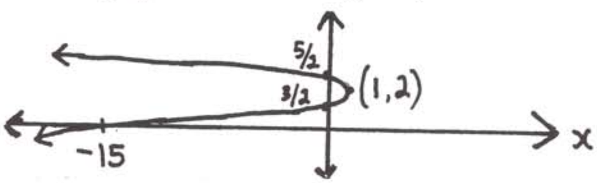
$-1 = -4(y-2)^2$

$\frac{1}{4} = (y-2)^2$

$\pm\frac{1}{2} = y - 2$

y-intercept: $\frac{3}{2}, \frac{5}{2}$

c) Use the information above to sketch the graph of $x - 1 = -4(y - 2)^2$.

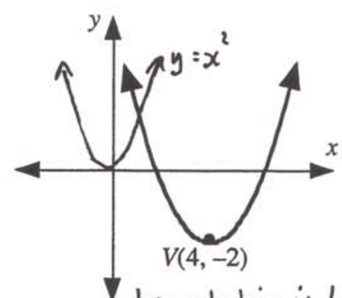


5. The graph shown is a transformed image of the graph of $y = x^2$. The transformation consists of a horizontal stretch by a factor of 2 about the y-axis, followed by a translation.

a) Determine the equation of the graph in general form.

$x \to \frac{1}{2}x \quad y = (\frac{1}{2}x)^2 \quad y = \frac{1}{4}x^2$

$\left.\begin{array}{l} x \to x-4 \\ y \to y+2 \end{array}\right\} \quad \underline{\underline{y + 2 = \frac{1}{4}(x-4)^2}}$



translation is 4 units right and 2 units down.

b) The given graph is stretched vertically by a factor of $\frac{1}{2}$ about the line $y = -4$.

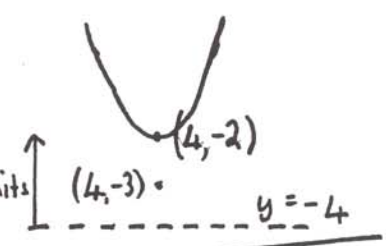
Determine the equation of the transformed graph in standard form.

the vertex is 2 units above the line $y = -4$ so after the the new vertex will be $\frac{1}{2}(2) = 1$ unit above $y = -4$ at $(4, -3)$

$h = 4 \quad k = -3$

From the equation $y - k = a(x-h)^2$ the vertical stretch by a factor of $\frac{1}{2}$ changes a from $\frac{1}{4}$ to $\frac{1}{2}(\frac{1}{4}) = \frac{1}{8}$

Equation is $\underline{\underline{y + 3 = \frac{1}{8}(x-4)^2}}$



or by formula

$y \to 2y \qquad 2y + 2 = \frac{1}{4}(x-4)^2$

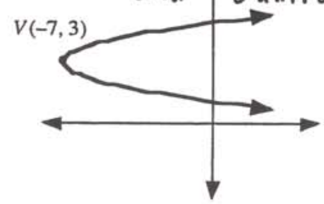
then $y \to y + (F-1)L \qquad y + 1 = \frac{1}{8}(x-4)^2$

$y \to y + (\frac{1}{2} - 1)(-4) \qquad y + 2 + 1 = \frac{1}{8}(x-4)^2$

$y \to y + 2 \qquad y + 3 = \frac{1}{8}(x-4)^2$

6. The graph shown is a transformed image of the graph of $x = y^2$. The transformation consists of a horizontal expansion by a factor of 2 about the y-axis, followed by a translation. Determine the equation of the graph in general form.

translation is 7 units left and 3 units up.



$x \to \frac{1}{2}x$ $\quad \frac{1}{2}x = y^2 \quad x = 2y^2$

$x \to x+7$ $\quad x+7 = 2(y-3)^2$

$y \to y-3$ $\quad x+7 = 2(y^2-6y+9)$

$x+7 = 2y^2-12y+18$

$\underline{\underline{2y^2 - x - 12y + 11 = 0}}$

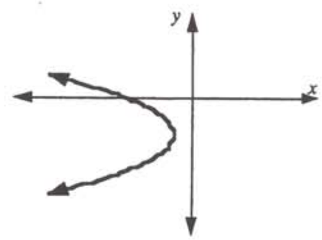
7. A graph has an equation of the form $x - h = 3(y - k)^2$. The vertex is $V(5, 0)$. If the graph is translated 3 units left and 2 units up, determine the equation of the transformed image in standard form.

$V(5, 0)$ $\quad x-5 = 3y^2$
h k

$x \to x+3$ $\quad x+3-5 = 3(y-2)^2$ $\quad \underline{\underline{x-2 = 3(y-2)^2}}$

$y \to y-2$

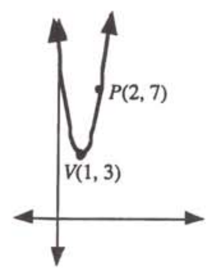
Multiple Choice 8. The graph shows a parabola with equation $x - h = a(y - k)^2$. How many of the parameters a, h, and k are positive?



A. 0
B. 1
C. 2
D. 3

$a < 0$ since graph opens left.
vertex (h, k) is in quadrant 3
so $h < 0$ and $k < 0$.

Numerical Response 9. The graph shown represents an equation of the form $y - k = a(x - h)^2$. If the vertex is $V(1, 3)$ and the graph passes through the point $P(2, 7)$, the value of a, to the nearest tenth is _____.



$y-3 = a(x-1)^2$ $\quad 7-3 = a(2-1)^2$
replace $(2, 7)$ $\quad 4 = a(1)$ $\quad a = 4$

(Record your answer in the numerical response box from left to right)

4	.	0	

Answer Key

1. **a)** $y = x^2 - 3$ **b)** $y = (x-5)^2$ **c)** $y = \frac{1}{16}x^2$ **d)** $y = \frac{1}{9}x^2$

2. **a)** horizontal stretch by a factor of 2 about the y-axis, then a translation 4 units right and 9 units down
b) **i)** $x \in \Re$ **ii)** $\{y \mid y \geq -9, y \in \Re\}$ **iii)** $(4, -9)$ **iv)** x-int $= -2$ and 10, y-int $= -5$

3. **a)** horizontal stretch by a factor of $1/\sqrt{2}$ about the y-axis, reflection in the x-axis, then a translation 6 units left. **b)** **i)** $x \in \Re$ **ii)** $\{y \mid y \leq 0, y \in \Re\}$ **iii)** $(-6, 0)$ **iv)** x-int $= -6$, y-int $= -72$

4. **a)** vertical stretch by a factor of $1/2$ about the x-axis, reflection in the y-axis, then a translation 1 unit right and 2 units up
b) **i)** $\{x \mid x \leq 1, x \in \Re\}$ **ii)** $y \in \Re$ **iii)** $(1, 2)$ **iv)** x-int $= -15$, y-int $= 1.5$ and 2.5

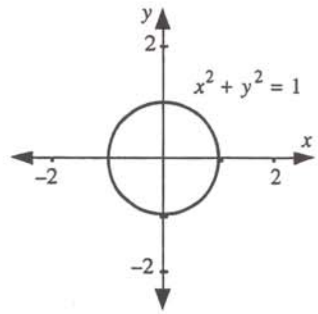
5. **a)** $x^2 - 8x - 4y + 8 = 0$ **b)** $y + 3 = 1/8(x-4)^2$

6. $2y^2 - x - 12y + 11 = 0$ **7.** $x - 2 = 3(y-2)^2$ **8.** A **9.**

4	.	0	

Conic Sections Lesson #5:
Transformations of Circles and Ellipses

Transformations of the Circle $x^2 + y^2 = 1$



The graph of the circle with equation $x^2 + y^2 = 1$ is shown.
Complete the following for the graph.

- domain $-1 \le x \le 1$
- range $-1 \le y \le 1$
- centre $(0,0)$
- radius 1

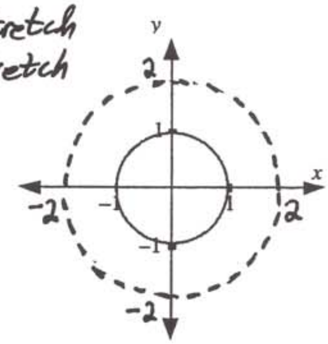
Part 1 *Transforming the Circle using Stretches*

x is replaced by $\frac{1}{2}x$ and y is replaced by $\frac{1}{2}y$ to get the equation

$$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{2}y\right)^2 = 1 \quad \text{or} \quad \frac{x^2}{2^2} + \frac{y^2}{2^2} = 1 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{4} = 1 \quad \text{or} \quad x^2 + y^2 = 4.$$

a) Complete the following for the graph of the equation $\frac{x^2}{4} + \frac{y^2}{4} = 1$.

- The transformation from $x^2 + y^2 = 1$ is a horizontal stretch by a factor of 2 about the y-axis and a vertical stretch by a factor of 2 about the x-axis.



b) Draw the transformed image on the grid and complete.

- domain $-2 \le x \le 2$
- range $-2 \le y \le 2$
- centre $(0,0)$
- radius 2

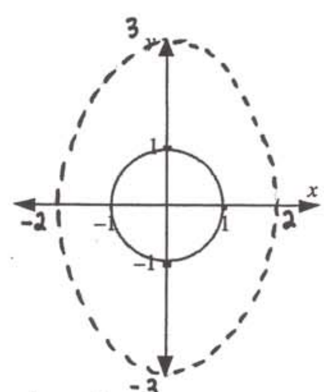
Part 2A *Transforming the Circle using Stretches*

x is replaced by $\frac{1}{2}x$ and y is replaced by $\frac{1}{4}y$ to get the equation

$$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{3}y\right)^2 = 1 \quad \text{or} \quad \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{9} = 1\,.$$

a) Complete for the graph of the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

- The transformation from $x^2 + y^2 = 1$ is a horizontal stretch by a factor of 2 about the y-axis and a vertical stretch by a factor of 3 about the x-axis.



b) Draw the transformed image and label the intercepts.

- domain $-2 \le x \le 2$
- range $-3 \le y \le 3$
- centre $(0,0)$
- length of horizontal axis (horizontal diameter) 4
- length of vertical axis (vertical diameter) 6

Part 2B *Transforming the Ellipse with Translations*

The ellipse in Part 2A is then transformed as follows:

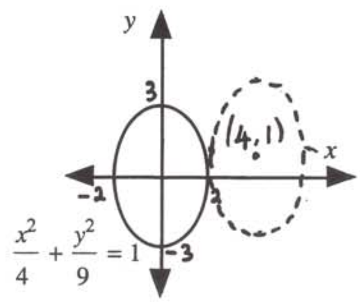
x is then replaced by $x-4$ and y is then replaced by $y-1$ to get the equation

$$\frac{(x-4)^2}{4}+\frac{(y-1)^2}{9}=1.$$

a) Complete for the graph of the equation $\frac{(x-4)^2}{4}+\frac{(y-1)^2}{9}=1.$

- The transformation from $\frac{x^2}{4}+\frac{y^2}{9}=1$ is a translation 4 units right and 1 unit up

b) Label the transformed image on the grid and complete.



- domain $2 \le x \le 6$ • range $-2 \le y \le 4$
- centre $(4, 1)$ • vertices $(4, -2)$ and $(4, 4)$
- length of major axis 6 • length of minor axis 4

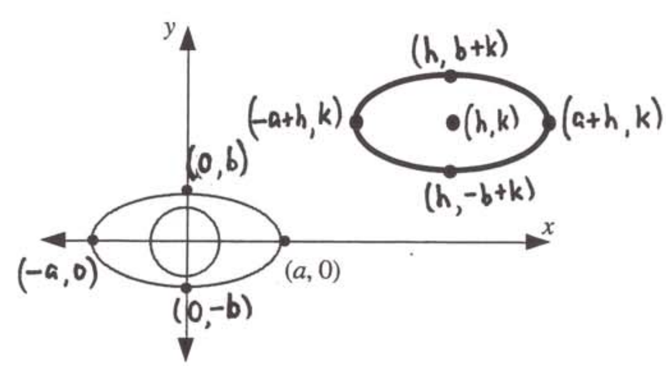
Part 3 *The General Case*

The circle $x^2+y^2=1$ is transformed into the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by a horizontal stretch about the y-axis by a factor of a and a vertical stretch about the x-axis by a factor of b.

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is transformed into the ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ by a translation h units right and k units up.

The diagram shows the circle and both ellipses

On the diagram label the coordinates of each point shown.



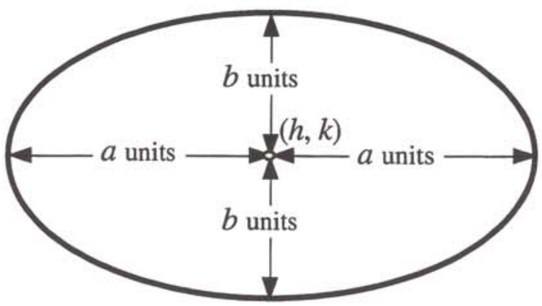
Complete for $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$

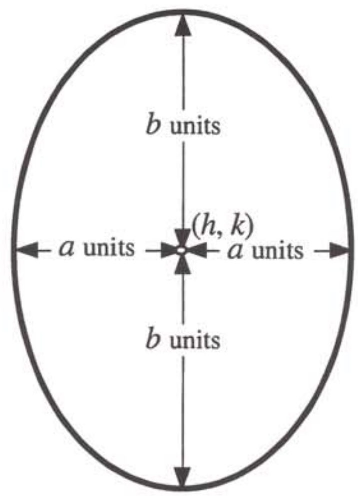
- domain $-a+h \le x \le a+h$
- range $-b+k \le y \le b+k$
- coordinates of centre (h, k)
- length of horizontal axis $2a$ • length of vertical axis $2b$

Features of the Graph of the Ellipse $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

The ellipse defined by the equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ has

- centre (h, k)
- domain $-a+h \le x \le a+h$
- range $-b+k \le y \le b+k$
- the length of the horizontal axis of $2a$
- the length of the vertical axis length of $2b$
- the **longer** of the two axes is called the **major** axis, the **shorter** is called the **minor** axis
- a is the horizontal stretch factor and b is the vertical stretch factor from the unit circle
- h is the horizontal translation and k is the vertical translation of the centre.



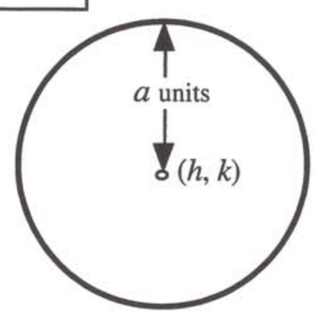


The circle can be considered as a special case of the ellipse with $b = a$.

Features of the Graph of the Circle $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{a^2}=1$ or $(x-h)^2+(y-k)^2=a^2$

The circle defined by the equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{a^2}=1$ has

- centre (h, k)
- radius a
- domain $-a+h \le x \le a+h$
- range $-a+k \le y \le a+k$



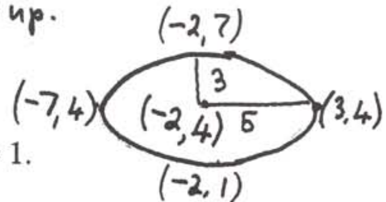
Consider the ellipse with equation $\frac{(x+2)^2}{25} + \frac{(y-4)^2}{9} = 1$.

a) Use transformations to describe how the graph of this conic section compares to the graph of the circle $x^2 + y^2 = 1$.

$x \to \frac{1}{5}x$ horizontal stretch by a factor of 5 about the y-axis

$y \to \frac{1}{3}y$ vertical stretch by a factor of 3 about the x-axis

$\left.\begin{array}{l} x \to x+2 \\ y \to y-4 \end{array}\right\}$ followed by a translation 2 units left and 4 units up.



b) Determine the following features of the graph of $\frac{(x+2)^2}{25} + \frac{(y-4)^2}{9} = 1$.

i) centre $(-2, 4)$

ii) horizontal length $2 \times 5 = 10$

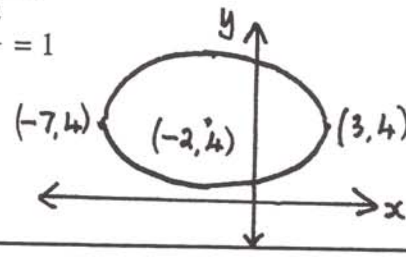
iii) vertical length $2 \times 3 = 6$

iv) domain $-7 \le x \le 3$

v) range $1 \le y \le 7$

vi) coordinates of vertices $(-7, 4)$ and $(3, 4)$

c) Use the information above to sketch the graph of $\frac{(x+2)^2}{25} + \frac{(y-4)^2}{9} = 1$



The graph of a circle with centre (0, 0) and radius 1 is transformed into an ellipse as shown.

a) Describe the series of transformations required to transform the graph of the circle into the graph of an ellipse with centre at (4, –6), a horizontal length of 4 units and a vertical length of 8 units.

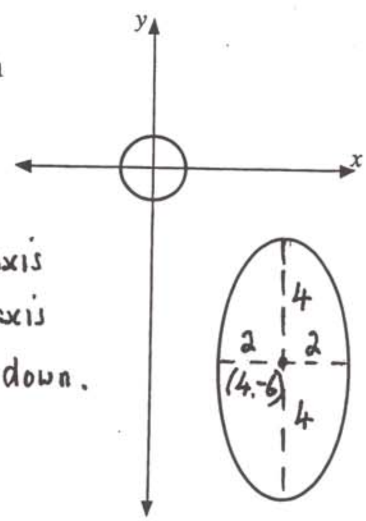
$2a = 4 \quad 2b = 8$

$a = 2 \quad b = 4$

horizontal stretch by a factor of 2 about the y-axis

vertical stretch by a factor of 4 about the x-axis

followed by a translation 4 units right and 6 units down.



b) Write the equation of the transformed image.

$x \to \frac{1}{2}x$

$y \to \frac{1}{4}y$

$x^2 + y^2 = 1$

$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{4}y\right)^2 = 1$

$\frac{x^2}{4} + \frac{y^2}{16} = 1$

$x \to x-4$

$y \to y+6$

$$\frac{(x-4)^2}{4} + \frac{(y+6)^2}{16} = 1$$

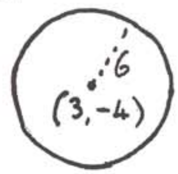
Consider the circle with equation $\frac{(x-3)^2}{36} + \frac{(y+4)^2}{36} = 1$.

a) Describe the series of transformations which would transform the graph $x^2 + y^2 = 1$ to the graph of $\frac{(x-3)^2}{36} + \frac{(y+4)^2}{36} = 1$.

$x \to \frac{1}{6}x$ horizontal stretch by a factor of 6 about the y-axis,
$y \to \frac{1}{6}y$ vertical stretch by a factor of 6 about the x-axis

followed by a translation 3 units right and 4 units down

$\left.\begin{matrix} x \to x-3 \\ y \to y+4 \end{matrix}\right\}$



b) Determine the following features of the graph of $\frac{(x-3)^2}{36} + \frac{(y+4)^2}{36} = 1$.

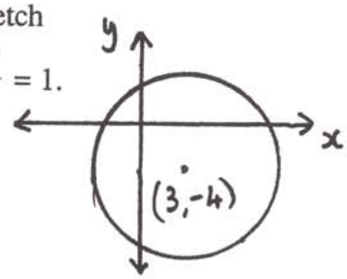
i) centre $(3, -4)$

ii) radius 6

iii) domain $-3 \le x \le 9$

iv) range $-10 \le y \le 2$

c) Use the information above to sketch the graph of $\frac{(x-3)^2}{36} + \frac{(y+4)^2}{36} = 1$.



In each case, write the equation of the circle in the form $(x-h)^2 + (y-k)^2 = r^2$ and in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1$.

a) centre (3,–7) and diameter 10

h, k radius 5
a = 5

$$\frac{(x-3)^2}{25} + \frac{(y+7)^2}{25} = 1$$

b) endpoints of a diameter at (4, 2) and (8, 2)

centre $(6, 2)$ radius $= 2$

$$\frac{(x-6)^2}{4} + \frac{(y-2)^2}{4} = 1$$

A translation of p units right and q units up can be described by the ordered pair (p, q).

a) Determine the equation of the circle $\frac{x^2}{100} + \frac{y^2}{100} = 1$ after a translation described by the ordered pair $(5, -2)$.

5 right $x \to x-5$

2 down $y \to y+2$

$$\frac{(x-5)^2}{100} + \frac{(y+2)^2}{100} = 1$$

b) If the point $P(6, 8)$ lies on the original circle, determine the coordinates of P', the image of P, under the transformation in a).

$(6+5, 8-2)$ $\quad (11, 6)$

c) If a point $Q(m, n)$ lies on the original circle, determine the coordinates of Q', the image of Q, under the transformation in a).

$(m+5, n-2)$

Complete Assignment Questions #1 - #17

Assignment

1. Determine the equation of the given conic after the transformation.

a) Determine the equation of the image of the circle $(x-2)^2 + (y-3)^2 = 1$ after a translation 2 units up and 3 units left.

$x \to x+3$, $y \to y-2$

$$((x+3)-2)^2 + ((y-2)-3)^2 = 1$$

$$\underline{\underline{(x+1)^2 + (y-5)^2 = 1}}$$

b) Determine the equation of the image of the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ after a horizontal stretch of factor 2 about the y-axis and a vertical stretch of factor $\frac{1}{2}$ about the x-axis.

$x \to \frac{1}{2}x$, $y \to 2y$

$$\frac{(\frac{1}{2}x)^2}{25} + \frac{(2y)^2}{4} = 1$$

$$\underline{\underline{\frac{x^2}{100} + y^2 = 1}}$$

c) Determine the equation of the image of the circle $(x-2)^2 + (y+5)^2 = 20$ under the translation defined by the mapping $(x, y) \to (x-2, y+5)$.

decrease x by 2 $\quad$ increase y by 5 $\quad$ 2 left, 5 up

$x \to x+2$

$y \to y-5$

$$(x+2-2)^2 + (y-5+5)^2 = 20$$

$$\underline{\underline{x^2 + y^2 = 20}}$$

2. Use transformations to describe how the graph of the given circle can be obtained from the graph of the circle $x^2 + y^2 = 1$.

a) $\dfrac{x^2}{16} + \dfrac{(y+4)^2}{9} = 1$ $x \to \frac{1}{4}x$ horizontal stretch by a factor of 4 about the y-axis

$\left(\frac{1}{4}x\right)^2 + \left(\frac{1}{3}y\right)^2 = 1$ $y \to \frac{1}{3}y$ vertical stretch by a factor of 3 about the x-axis

$\frac{x^2}{16} + \frac{y^2}{9} = 1$ $y \to y+4$ followed by a translation 4 units down.

$\frac{x^2}{16} + \frac{(y+4)^2}{9} = 1$

b) $4x^2 + 4y^2 = 1$ $x \to 2x$ horizontal stretch by a factor of $\frac{1}{2}$ about the y-axis

$(2x)^2 + (2y)^2 = 1$ $y \to 2y$ vertical stretch by a factor of $\frac{1}{2}$ about the x-axis

$4x^2 + 4y^2 = 1$

c) $\dfrac{25(x-1)^2}{16} + \dfrac{(y+3)^2}{121} = 1$ horizontal stretch by a factor of $\frac{4}{5}$ about the y-axis

$\left(\frac{5}{4}x\right)^2 + \left(\frac{1}{11}y\right)^2 = 1$ $x \to \frac{5}{4}x$ vertical stretch by a factor of 11 about the x-axis

$\frac{25x^2}{16} + \frac{y^2}{121} = 1$ $y \to \frac{1}{11}y$ followed by a translation 1 unit right and 3 units down.

$\frac{25(x-1)^2}{16} + \frac{(y+3)^2}{121} = 1$ $\left.\begin{array}{l} x \to x-1 \\ y \to y+3 \end{array}\right\}$

d) $9x^2 + 9(y-8)^2 = 4$ $\frac{9}{4}x^2 + \frac{9}{4}(y-8)^2 = 1$

$\left(\frac{3}{2}x\right)^2 + \left(\frac{3}{2}y\right)^2 = 1$ $x \to \frac{3}{2}x$ horizontal stretch by a factor of $\frac{2}{3}$ about the y-axis

$\frac{9}{4}x^2 + \frac{9}{4}y^2 = 1$ $y \to \frac{3}{2}y$ vertical stretch by a factor of $\frac{2}{3}$ about the x-axis

$\frac{9}{4}x^2 + \frac{9}{4}(y-8)^2 = 1$ $y \to y-8$ followed by a translation 8 units up.

$9x^2 + 9(y-8)^2 = 4$

3. A translation of p units right and q units up can be described by the ordered pair (p, q).

a) Determine the equation of the circle $(x-2)^2 + y^2 = 25$ after a translation described by the ordered pair $(-3, 4)$.

3 left $x \to x+3$ $(x+3-2)^2 + (y-4)^2 = 25$ $(x+1)^2 + (y-4)^2 = 25$

4 up $y \to y-4$

b) If the point $P(5, 4)$ lies on the original circle, determine the coordinates of P', the image of P, under the transformation in a).

3 left $(5-3, 4+4)$ $(2, 8)$

4 up

c) If a point $Q(m, n)$ lies on the original circle, determine the coordinates of Q', the image of Q, under the transformation in a).

$(m-3, n+4)$

4. Consider the ellipse with equation $(x+2)^2 + \frac{(y+1)^2}{4} = 1$. $a = 1$ $b = 2$ centre $(-2, -1)$

a) Use transformations to describe how the graph of this conic section compares to the graph of the circle $x^2 + y^2 = 1$.

$y \to \frac{1}{2}y$ vertical stretch by a factor of 2 about the x-axis followed by a translation 2 units left and 1 unit down.

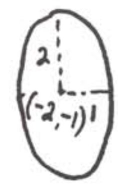
$x \to x+2$

$y \to y+1$

$x^2 + \left(\frac{1}{2}y\right)^2 = 1$

$x^2 + \frac{y^2}{4} = 1$

$(x+2)^2 + \frac{(y+1)^2}{4} = 1$



b) Determine the following features of the graph of $(x+2)^2 + \frac{(y+1)^2}{4} = 1$.

i) centre $(-2, -1)$

ii) horizontal length $2 \times 1 = 2$

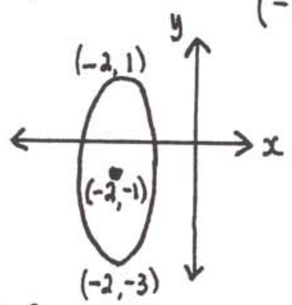
iii) vertical length $2 \times 2 = 4$

iv) domain $-3 \le x \le -1$

v) range $-3 \le y \le 1$

vi) coordinates of vertices $(-2, -3)$ and $(-2, 1)$

c) Use the information above to sketch the graph of $(x+2)^2 + \frac{(y+1)^2}{4} = 1$.



5. Consider the circle with equation $x^2 + (y-12)^2 = 81$.

a) Describe the series of transformations which would transform the graph $x^2 + y^2 = 1$ to the graph of $x^2 + (y-12)^2 = 81$. $\frac{x^2}{81} + \frac{(y-12)^2}{81} = 1$

$x \to \frac{1}{9}x$ horizontal stretch by a factor of 9 about the y-axis

$y \to \frac{1}{9}y$ vertical stretch by a factor of 9 about the x-axis

$y \to y-12$ followed by a translation 12 units up.

$\left(\frac{1}{9}x\right)^2 + \left(\frac{1}{9}y\right)^2 = 1$

$\frac{x^2}{81} + \frac{y^2}{81} = 1$

$\frac{x^2}{81} + \frac{(y-12)^2}{81} = 1$

b) Determine the following features of the graph of $x^2 + (y-12)^2 = 81$.

i) centre $(0, 12)$

ii) radius 9

iii) domain $-9 \le x \le 9$

iv) range $3 \le y \le 21$

v) x- and y-intercepts

x-int. $y = 0$

$x^2 + (-12)^2 = 81$

$x^2 + 144 = 81$

$x^2 = -63$

no x-intercepts

y-int. $x = 0$

$(y-12)^2 = 81$

$y - 12 = \pm 9$

$y = \pm 9 + 12$

y-intercepts 3, 21

c) Use the information in a) and b) to sketch the graph of $x^2 + (y - 12)^2 = 81$. $\frac{x^2}{81} + \frac{(y-12)^2}{81} = 1$

d) The graph of $x^2 + (y - 12)^2 = 81$ is stretched horizontally by a factor of $\frac{2}{3}$ about the line $x = 6$.

graph is transformed to an ellipse.

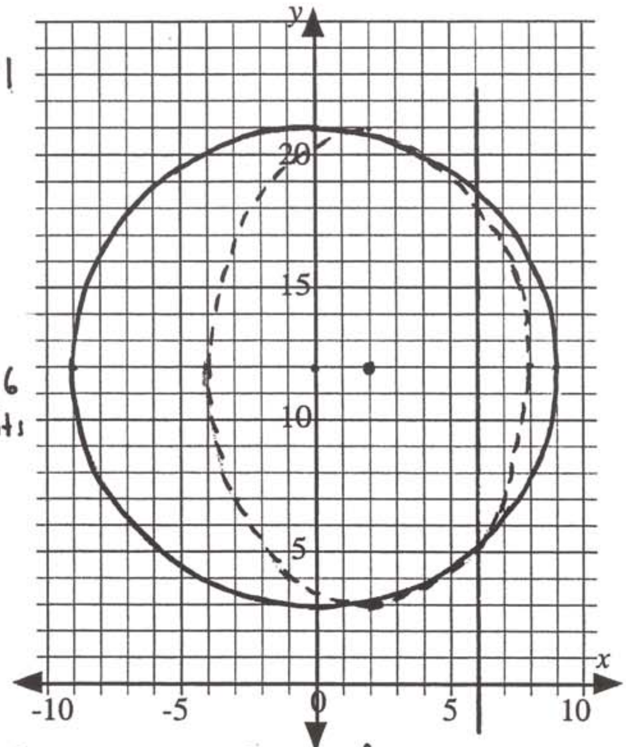
Sketch the transformed graph and determine its equation.

Centre of circle $(0,12)$ is 6 units left of $x = 6$

After ~~stretch~~ the centre is $\frac{2}{3}(6) = 4$ units left of $x = 6$ at $(2,12)$

Horizontal length of ellipse $= 12$ $\quad 2a = 12$, $a = 6$

Vertical length of ellipse $= 18$ $\quad 2b = 18$, $b = 9$



Equation of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$$\frac{(x-2)^2}{36} + \frac{(y-12)^2}{81} = 1$$

or by formula

$x \to \frac{3}{2}x$

$x \to x + (F-1)L$

$x \to x + (\frac{2}{3} - 1)(6)$

$x \to x - 2$

$x^2 + (y-12)^2 = 81$

$(\frac{3}{2}x)^2 + (y-12)^2 = 81$, $\frac{9}{4}x^2 + (y-12)^2 = 81$

$\frac{9}{4}(x-2)^2 + (y-12)^2 = 81$ divide by 81

$\frac{(x-2)^2}{36} + \frac{(y-12)^2}{81} = 1$

6. In each case, write the equation of the circle in the form $(x - h)^2 + (y - k)^2 = r^2$.

a) centre $(-8, 11)$ and radius $\frac{5}{3}$

h, k

$(x+8)^2 + (y-11)^2 = (\frac{5}{3})^2$

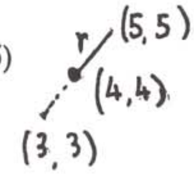
$$(x+8)^2 + (y-11)^2 = \frac{25}{9}$$

b) endpoints of a diameter at $(3, 3)$ and $(5, 5)$

centre $(4,4)$ $\quad r^2 = (5-4)^2 + (5-4)^2$

$r^2 = 1 + 1 = 2$

$$(x-4)^2 + (y-4)^2 = 2$$



7. In each case, write the equation of the circle in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1$.

a) centre $(6, -1)$ and passing through $(0, 7)$

$r^2 = (0-6)^2 + (7+1)^2$

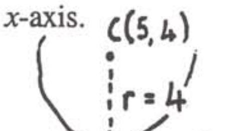
$r^2 = 36 + 64 = 100$

$$\frac{(x-6)^2}{100} + \frac{(y+1)^2}{100} = 1$$

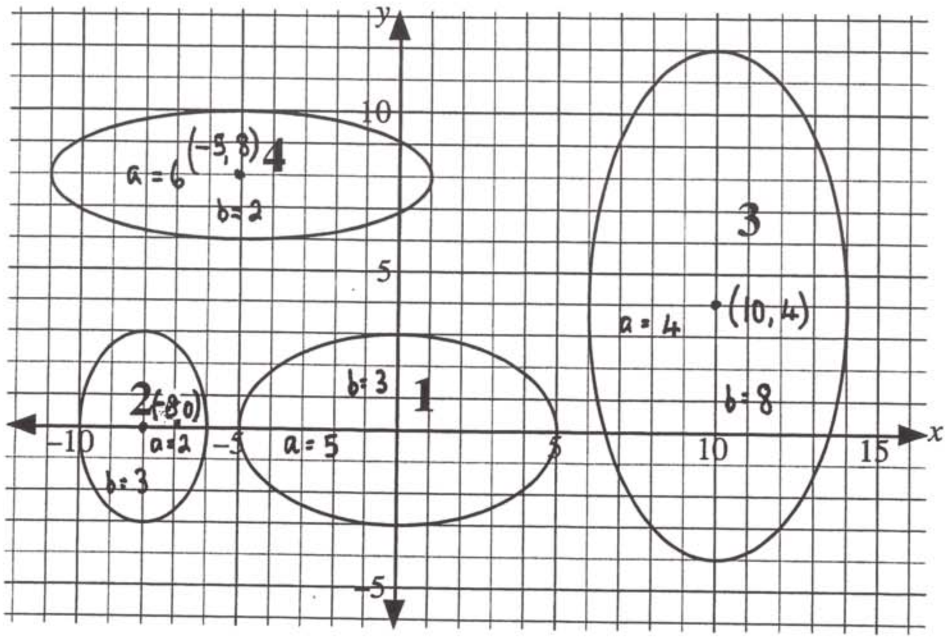
b) centre $(5, 4)$ and tangent to the x-axis.

$r = 4$

$$\frac{(x-5)^2}{16} + \frac{(y-4)^2}{16} = 1$$



8. Consider the following ellipses



In each case describe the transformations that have been applied to the unit circle $x^2 + y^2 = 1$ to produce the ellipse.
Express the equation of each ellipse in standard form.

1. centre $(0, 0)$ $\quad x \to \frac{1}{5}x$ horizontal stretch by a factor of 5 about the y-axis $\quad \frac{x^2}{25} + \frac{y^2}{9} = 1$
   $a = 5$, $b = 3$ $\quad y \to \frac{1}{3}y$ vertical stretch by a factor of 3 about the x-axis

2. centre $(-8, 0)$ $\quad x \to \frac{1}{2}x$ horizontal stretch by a factor of 2 about the y-axis $\quad \frac{x^2}{4} + \frac{y^2}{9} = 1$
   $a = 2$, $b = 3$ $\quad y \to \frac{1}{3}y$ vertical stretch by a factor of 3 about the x-axis
   $x \to x + 8$ followed by a translation 8 units left. $\quad \frac{(x+8)^2}{4} + \frac{y^2}{9} = 1$

3. centre $(10, 4)$ $\quad x \to \frac{1}{4}x$ horizontal stretch by a factor of 4 about the y-axis $\quad \frac{x^2}{16} + \frac{y^2}{64} = 1$
   $a = 4$, $b = 8$ $\quad y \to \frac{1}{8}y$ vertical stretch by a factor of 8 about the x-axis
   $x \to x - 10$ followed by a translation 10 units right and $\quad \frac{(x-10)^2}{16} + \frac{(y-4)^2}{64} = 1$
   $y \to y - 4$ 4 units up.

4. centre $(-5, 8)$ $\quad x \to \frac{1}{6}x$ horizontal stretch by a factor of 6 about the y-axis $\quad \frac{x^2}{36} + \frac{y^2}{4} = 1$
   $a = 6$, $b = 2$ $\quad y \to \frac{1}{2}y$ vertical stretch by a factor of 2 about the x-axis
   $x \to x + 5$ followed by a translation 5 units left and $\quad \frac{(x+5)^2}{36} + \frac{(y-8)^2}{4} = 1$
   $y \to y - 8$ 8 units up.

9. Determine the equation of each of the following ellipses.

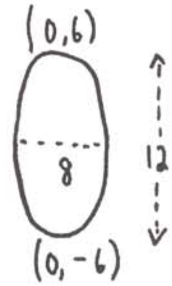
a) Vertices are (0, –6) and (0, 6) and horizontal length is 8 units.

centre $(0,0)$

$2a = 8,\ a = 4$

$2b = 12,\ b = 6$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1$$



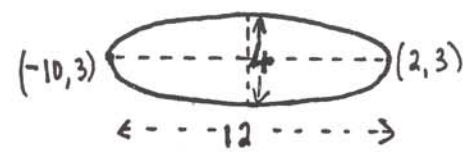
b) Vertices are (–10, 3) and (2, 3) and vertical length is 4 units.

Centre $(-4, 3)$

$2a = 12,\ a = 6$

$2b = 4,\ b = 2$

$$\frac{(x+4)^2}{36} + \frac{(y-3)^2}{4} = 1$$



10. a) Sketch the graph of the circle with equation $x^2 + y^2 = 9$.

b) Sketch the graph after a vertical expansion by a factor of 2 about the line $y = -3$.

c) Determine the equation of the transformed graph.

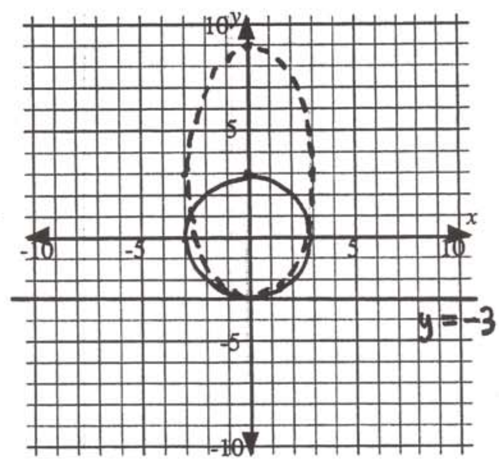
Centre of the circle $(0,0)$ is 3 units above $y = -3$.

After the stretch the centre of the ellipse is $2(3) = 6$ units above $y = -3$ at $(0, 3)$

Minor axis $= 6$ so $a = 3$

Major axis $= 12$ so $b = 6$

$$\frac{x^2}{9} + \frac{(y-3)^2}{36} = 1$$



11. a) Determine the equation of an ellipse with centre (1, –2) , horizontal length of 6 units, and vertical length of 4 units.

$b = 2$ $\quad a = 3$

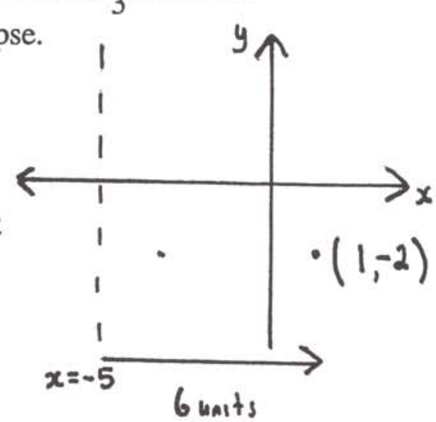
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

b) The ellipse in a) undergoes a horizontal compression by a factor of $\frac{1}{3}$ about the line $x = -5$. Determine the equation of the transformed ellipse.

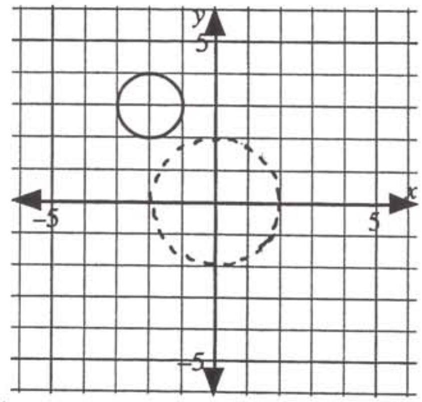
Centre $(1, -2)$ is 6 units right of $x = -5$ so after the stretch the centre will be $\frac{1}{3}(6) = 2$ units right of $x = -5$ at $(-3, -2)$

The horizontal stretch by $\frac{1}{3}$ changes the a value from 3 to $\frac{1}{3}(3) = 1$. b is unchanged.

$$(x+3)^2 + \frac{(y+2)^2}{4} = 1$$



12. Describe the series of transformations which would transform the graph $x^2 + y^2 = 4$ to the given graph and write the equation of the given graph in general form.



given graph
centre $(-2, 3)$
radius = 1

radius of $x^2+y^2=4$ is 2. centre $(0,0)$

$x \to 2x$ horizontal stretch by a factor of $\frac{1}{2}$ about the y-axis

$y \to 2y$ vertical stretch by a factor of $\frac{1}{2}$ about the x-axis

$x \to x+2$ followed by a translation 2 units left

$y \to y-3$ and 3 units up.

stretch	translation
$x^2+y^2=4$	$x^2+y^2=1$
$(2x)^2+(2y)^2=4$	$(x+2)^2+(y-3)^2=1$
$4x^2+4y^2=4$	$x^2+4x+4+y^2-6y+9=1$
$x^2+y^2=1$	$\underline{\underline{x^2+y^2+4x-6y+12=0}}$

Multiple Choice

13. Which of the following circles has a diameter of $\sqrt{20}$?

A. $(x-2)^2 + (y-2)^2 = \sqrt{5}$ $\quad r^2=\sqrt{5} \quad r=5^{1/4} \quad d=2(5^{1/4}) \neq \sqrt{20}$

B. $4x^2 + 4y^2 = 20$ $\quad x^2+y^2=5 \quad r^2=5 \quad r=\sqrt{5} \quad d=2\sqrt{5}=\sqrt{4}\sqrt{5}=\sqrt{20}$

C. $\frac{x^2}{4} + \frac{(y-3)^2}{4} = \frac{5}{2}$ $\quad x^2+(y-3)^2=10 \quad r^2=10 \quad r=\sqrt{10} \quad d=2\sqrt{10} \neq \sqrt{20}$

D. $x^2 = 20 - y^2$ $\quad x^2+y^2=20 \quad r^2=20 \quad r=\sqrt{20} \quad d=2\sqrt{20} \neq \sqrt{20}$

14. The diagram shows the ellipse with equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

The value of $a + b$ is

A. 20

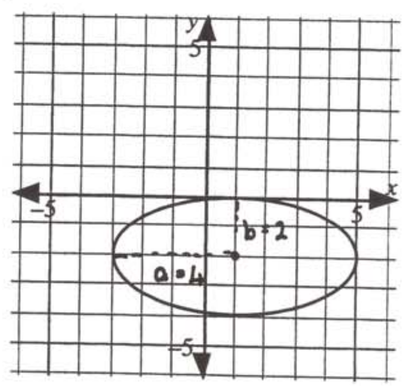
B. 12

C. 6

D. -1

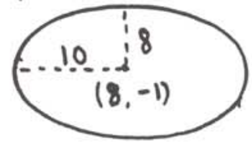
length of major axis = 8
$2a=8 \quad a=4$
length of minor axis = 4
$2b=4 \quad b=2$

$a+b=4+2=6$



15. Consider the ellipse with equation $\frac{(x-8)^2}{100} + \frac{(y+1)^2}{64} = 1$. Which one of the following statements is **false**?

$a = 10$ $\quad b = 8$ $\quad$ centre $(8, -1)$



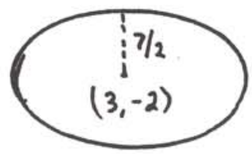
A. The ellipse contains points in all four quadrants. ✓
B. The line $x = 8$ is an axis of symmetry. ✓
C. The centre lies in quadrant 4. ✓
D. The ellipse is a vertical ellipse. ×

Numerical Response

16. Consider the quadratic relation with equation $\frac{(x-3)^2}{81} + \frac{4(y+2)^2}{49} = 1$.
The maximum y coordinate, to the nearest tenth, on the graph of the relation is _____ .
(Record your answer in the numerical response box from left to right)

1	.	5	

centre $(3, -2)$
$a^2 = 81 \quad a = 9$
$b^2 = \frac{49}{4} \quad b = \frac{7}{2}$



$\max y = -2 + \frac{7}{2}$
$= \frac{3}{2}$

17. The circle of the equation $5x^2 + 5y^2 = 1$ is a transformed image of the circle with equation $x^2 + y^2 = 1$. The scale factor, to the nearest hundredth, of the horizontal and vertical stretch is _____ .
(Record your answer in the numerical response box from left to right)

0	.	4	5

$y \to \sqrt{5}y \qquad$ scale factor $= \frac{1}{\sqrt{5}} = 0.45$

Answer Key

1. a) $(x+1)^2 + (y-5)^2 = 1$ **b)** $\frac{x^2}{100} + y^2 = 1$ **c)** $x^2 + y^2 = 20$

2. a) a horizontal stretch by a factor of 4 about the y-axis, a vertical stretch by a factor of 3 about the x-axis, followed by a translation 4 units down.

b) horizontal stretch about the y-axis and vertical stretch about the x-axis by a factor of $\frac{1}{2}$.

c) a horizontal stretch by a factor of $\frac{4}{5}$ about the y-axis, a vertical stretch by a factor of 11 about the x-axis, followed by a translation 1 unit right and 3 units down.

d) horizontal stretch about the y-axis and vertical stretch about the x-axis by a factor of $\frac{2}{3}$ followed by a vertical translation 8 units up.

3. a) $(x+1)^2 + (y-4)^2 = 25$ **b)** $(2, 8)$ **c)** $(m-3, n+4)$

4. a) vertical stretch by a factor of 2 about the x-axis, followed by a translation 2 units left and 1 unit down

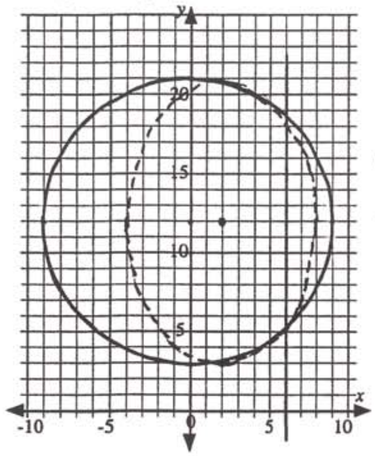
b) i) $(-2, -1)$ **ii)** 2 **iii)** 4 **iv)** $\{x \mid -3 \le x \le -1, x \in \Re\}$
v) $\{y \mid -3 \le y \le 1, y \in \Re\}$ **vi)** $(-2, -3)$ and $(-2, 1)$

5. a) horizontal stretch about the y-axis by a factor of 9 and vertical stretch about the x-axis by a factor of 9, followed by a translation 12 units up.

b) i) (0, 12) **ii)** 9 **iii)** $\{x \mid -9 \leq x \leq 9, x \in \Re\}$ **iv)** $\{y \mid 3 \leq y \leq 21, y \in \Re\}$

v) x-int = none, y-int = 3 and 21

c) see graph below



d) $\dfrac{(x-2)^2}{36} + \dfrac{(y-12)^2}{81} = 1$

6. a) $(x+8)^2 + (y-11)^2 = \frac{25}{9}$ **b)** $(x-4)^2 + (y-4)^2 = 2$

7. a) $\dfrac{(x-6)^2}{100} + \dfrac{(y+1)^2}{100} = 1$ **b)** $\dfrac{(x-5)^2}{16} + \dfrac{(y-4)^2}{16} = 1$

8. ellipse 1 a horizontal stretch by a factor of 5 about the y-axis and a vertical stretch by a factor of 3 about the x-axis: $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

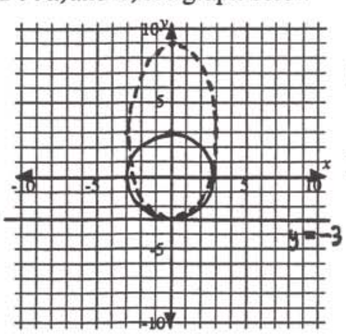
ellipse 2 a horizontal stretch by a factor of 2 about the y-axis, a vertical stretch by a factor of 3 about the x-axis, followed by a horizontal translation 8 units left: $\dfrac{(x+8)^2}{4} + \dfrac{y^2}{9} = 1$

ellipse 3 a horizontal stretch by a factor of 4 about the y-axis, a vertical stretch by a factor of 8 about the x-axis, then a translation 10 units right and 4 units up: $\dfrac{(x-10)^2}{16} + \dfrac{(y-4)^2}{64} = 1$

ellipse 4 a horizontal stretch by a factor of 6 about the y-axis, a vertical stretch by a factor of 2 about the x-axis, then a translation 5 units left and 8 units up: $\dfrac{(x+5)^2}{36} + \dfrac{(y-8)^2}{4} = 1$

9. a) $\dfrac{x^2}{16} + \dfrac{y^2}{36} = 1$ **b)** $\dfrac{(x+4)^2}{36} + \dfrac{(y-3)^2}{4} = 1$

10. a) and **b)** see graph below



11. a) $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{4} = 1$ **b)** $(x+3)^2 + \dfrac{(y+2)^2}{4} = 1$

12. horizontal stretch about the y-axis by a factor of $\frac{1}{2}$, and a vertical stretch about the x-axis by a factor of $\frac{1}{2}$, followed by a translation 2 units left and 3 units up. $x^2 + y^2 + 4x - 6y + 12 = 0$

13. B **14.** C **15.** D

16.

1	.	5	

17.

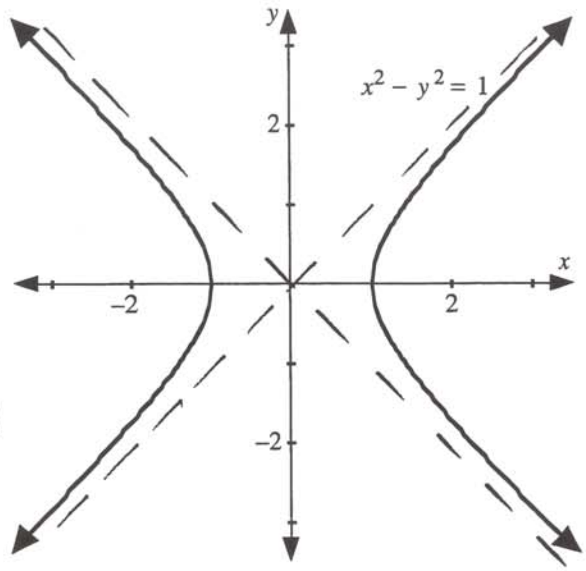
0	.	4	5

Conic Sections Lesson #6: Transformations of Hyperbolas

Transformations of the Hyperbola $x^2 - y^2 = 1$

The graph of the hyperbola with equation $x^2 - y^2 = 1$ is shown.
The curve has asymptotes with slopes of ± 1.
Sketch the asymptotes on the grid.
Complete the following for the graph.

- domain $x \le -1$ or $x \ge 1$
- range $y \in R$
- coordinates of centre $(0,0)$
- coordinates of vertices $(-1,0)$ and $(1,0)$
- the length of the transverse axis (i.e. distance between vertices) 2



Part 1A *Transforming the Hyperbola using Stretches*

x is replaced by $\frac{1}{2}x$ and y is replaced by $\frac{1}{3}y$ to get the equation

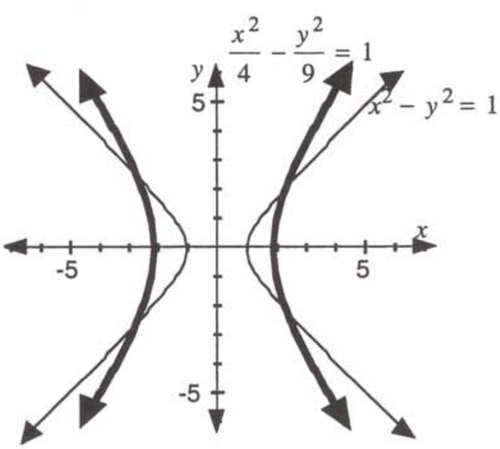
$$\left(\frac{1}{2}x\right)^2 - \left(\frac{1}{3}y\right)^2 = 1 \qquad \text{or} \qquad \frac{x^2}{4} - \frac{y^2}{9} = 1.$$

a) Complete the following for the graph of the equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

- The transformation from $x^2 - y^2 = 1$ is a horizontal stretch by a factor of 2 about the y-axis and a vertical stretch by a factor of 3 about the x-axis.

b) The transformed image is shown on the grid. Complete the following.

- domain $x \le -2$ or $x \ge 2$
- range $y \in R$
- coordinates of centre $(0,0)$
- coordinates of vertices $(-2,0)$ and $(2,0)$
- distance between vertices 4



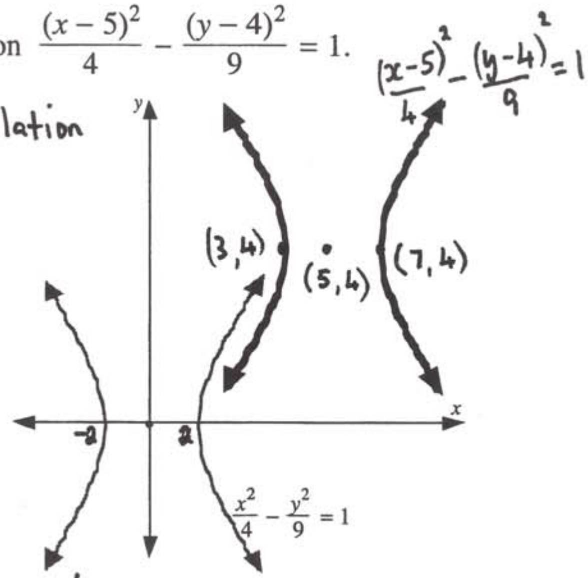
Note that the slopes of the asymptotes are now $\pm\frac{3}{2}$.

Part 1B ***Transforming the Hyperbola with Stretches and Translations***

x is then replaced by $x-5$ and y is then replaced by $y-4$ to get the equation
$\frac{(x-5)^2}{4} - \frac{(y-4)^2}{9} = 1$.

a) Complete the following for the graph of the equation $\frac{(x-5)^2}{4} - \frac{(y-4)^2}{9} = 1$.

- The transformation from $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is a translation 5 units right and 4 units up.



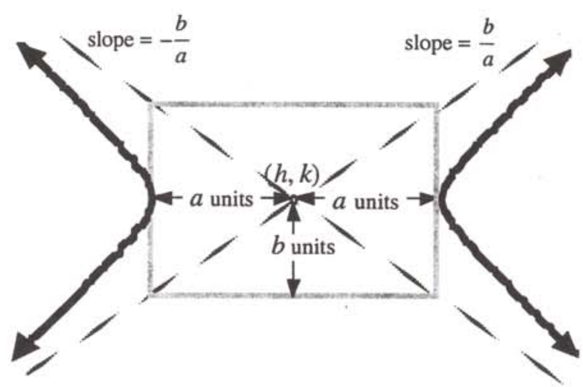
b) Label the transformed image which is shown on the grid and complete.

- domain $x \leq 3$ or $x \geq 7$
- range $y \in R$
- centre $(5, 4)$
- vertices $(3, 4)$ and $(7, 4)$
- distance between vertices (length of transverse axis) 4

Features of the Graph of the Hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

The hyperbola defined by the equation
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ **opens along the horizontal axis** and has:

- centre (h, k)
- domain: $x \leq -a + h$ or $x \geq a + h$
- range: $y \in \Re$
- vertices $(-a + h, k)$ and $(a + h, k)$
- the transverse axis is the line joining the vertices and its length is equal to $2a$
- the conjugate axis passes through the centre perpendicular to the transverse axis and its length is equal to $2b$



- asymptote with slopes $\pm\frac{b}{a}$
- a is the horizontal stretch factor and b is the vertical stretch factor from $x^2 - y^2 = 1$
- h is the horizontal translation and k is the vertical translation of the centre.

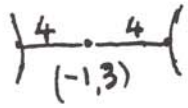
Consider the hyperbola with equation $\frac{(x+1)^2}{16} - \frac{9(y-3)^2}{25} = 1$. $a = 4 \quad b = 5/3$
$h = -1 \quad k = 3$

a) Use transformations to describe how the graph of this conic section can be obtained from the graph of the hyperbola $x^2 - y^2 = 1$. horizontal stretch by a factor of 4 about the y-axis, a vertical stretch by a factor of $\frac{5}{3}$ about the x-axis, followed by a translation 1 unit left and 3 units up.

$x \to \frac{1}{4}x \quad y \to \frac{3}{5}y \qquad x^2 - y^2 = 1 \quad \left(\frac{1}{4}x\right)^2 - \left(\frac{3}{5}y\right)^2 = 1 \quad \frac{x^2}{16} - \frac{9y^2}{25} = 1$

$x \to x+1 \quad y \to y-3 \qquad \frac{(x+1)^2}{16} - \frac{9(y-3)^2}{25} = 1$

b) Determine the following features of the graph of $\frac{(x+1)^2}{16} - \frac{9(y-3)^2}{25} = 1$.



i) centre
$(-1, 3)$

ii) distance between vertices
$2 \times 4 = 8$

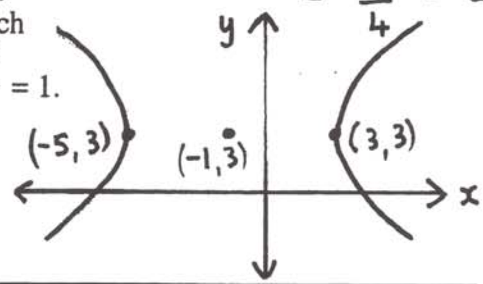
iii) coordinates of vertices
$(-5, 3)$ and $(3, 3)$

iv) domain
$x \leq -5$ or $x \geq 3$

v) range
$y \in R$

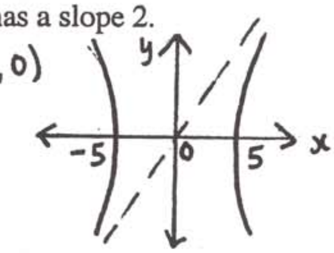
vi) slopes of the asymptotes
$\pm \frac{5/3}{4} = \pm \frac{5}{12}$

c) Use the information above to sketch the graph of $\frac{(x+1)^2}{16} - \frac{9(y-3)^2}{25} = 1$.



A hyperbola has vertices at (5, 0) and (–5, 0). One of the asymptotes has a slope 2.

a) Find the equation of the hyperbola in standard form.   centre$(0,0)$

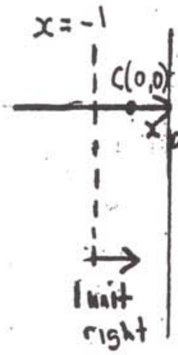


$2a = 10 \quad a = 5$ slope $= 2 = \frac{b}{a}$ $2a = b$ $b = 10$

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$$\underline{\frac{x^2}{25} - \frac{y^2}{100} = 1}$$

b) The hyperbola in a) is transformed by a horizontal stretch by a factor of 4 about the line $x = -1$. Determine the equation of the transformed hyperbola.



the centre $(0,0)$ is 1 unit right of $x = -1$.
after the horizontal stretch factor 4 the centre is $4(1) = 4$ units right of $x = -1$ at $(3,0)$
Horizontal stretch factor 4 means a changes to $4(5) = 20$
$a^2 = 400$
b is unchanged.

Equation is $\underline{\frac{(x-3)^2}{400} - \frac{y^2}{100} = 1}$

OR

In the transformation unit we had $x \to \frac{1}{4}x$ followed by $x \to x + (F-1)L$ ie $x \to x + (4-1)$
$x \to x - 3$

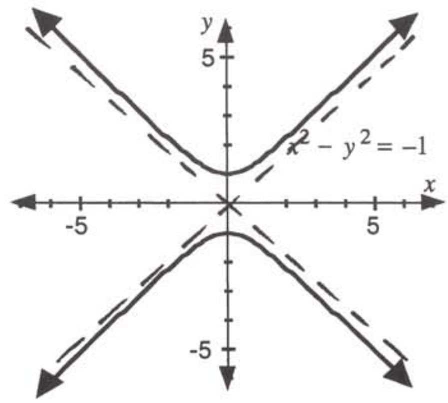
$\frac{x^2}{25} - \frac{y^2}{100} = 1$

$\frac{\left(\frac{1}{4}x\right)^2}{25} - \frac{y^2}{100} = 1$

$\frac{x^2}{400} - \frac{y^2}{100} = 1. \quad \frac{(x-3)^2}{400} - \frac{y^2}{100} =$

Transformations of the Hyperbola $x^2 - y^2 = -1$

The graph of the hyperbola with equation $x^2 - y^2 = -1$ is shown. The curve has asymptotes with slopes of ± 1.



Sketch the asymptotes on the grid.

Complete the following for the graph.

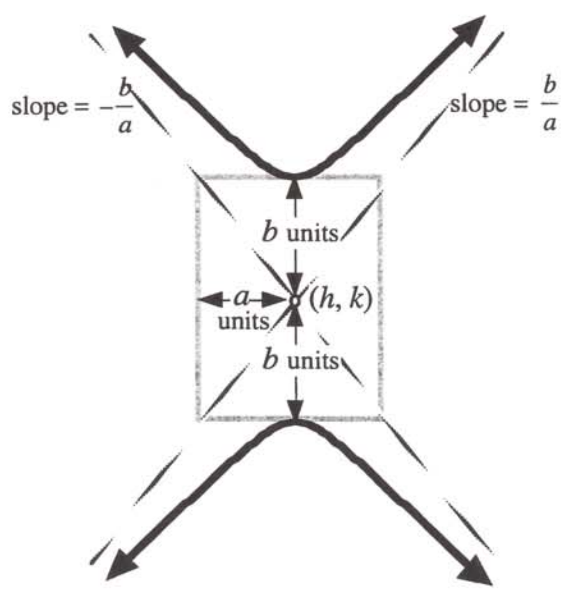
- domain: $x \in R$ • range $y \le -1$ or $y \ge 1$
- centre $(0,0)$ • vertices $(0,-1)$ and $(0,1)$
- the length of the transverse axis (i.e. distance between vertices) 2

Features of the Graph of the Hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$

The hyperbola defined by the equation

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$ **opens along the vertical axis** and has

- centre (h, k)
- domain: $x \in \Re$
- range: $y \le -b + k$ or $y \ge b + k$
- the transverse axis is the line joining the vertices and its length is equal to $2b$
- the conjugate axis passes through the centre perpendicular to the transverse axis and its length is equal to $2a$
- asymptote with slopes $\pm\frac{b}{a}$
- a is the horizontal stretch factor and b is the vertical stretch factor from $x^2 - y^2 = -1$.



- h is the horizontal translation and k is the vertical translation of the centre.

Consider the hyperbola with equation $9(x+1)^2 - \frac{(y+2)^2}{9} = -1$. $\frac{(x+1)^2}{1/9} - \frac{(y+2)^2}{9} = -1$

$a = \frac{1}{3}$ $b = 3$ $h = -1$ $k = -2$

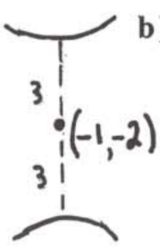
a) Use transformations to describe how the graph of this conic section can be obtained from the graph of the hyperbola $x^2 - y^2 = -1$.

$\left.\begin{array}{l} x \to 3x \\ y \to \frac{1}{3}y \end{array}\right\}$ $x^2 - y^2 = -1$ $\quad (3x)^2 - (\frac{1}{3}y)^2 = -1$

$9x^2 - \frac{y^2}{9} = -1$

$\left.\begin{array}{l} x \to x+1 \\ y \to y+2 \end{array}\right\}$ $9(x+1)^2 - \frac{(y+2)^2}{9} = -1$

horizontal stretch by a factor of $\frac{1}{3}$ about the y-axis, vertical stretch by a factor of 3 about the x-axis followed by a translation 1 unit left and 2 units down.



b) Determine the following features of the graph of $9(x+1)^2 - \frac{(y+2)^2}{9} = -1$.

i) centre

$(-1, -2)$

ii) distance between vertices

$2 \times 3 = 6$

iii) coordinates of vertices

$(-1, -5)$ and $(-1, 1)$

iv) domain

$x \in R$

v) range

$y \leq -5$ or $y \geq 1$

vi) slopes of the asymptotes

$\pm \frac{b}{a} = \pm \frac{3}{1/3} = \pm 9$

vii) x- and y-intercepts (to nearest tenth).

x-int $y = 0$

$9(x+1)^2 - \frac{4}{9} = -1$

$9(x+1)^2 = -\frac{5}{9}$

$(x+1)^2 = \frac{-5}{81}$

no x-intercepts

y-int $x = 0$

$9(1)^2 - \frac{(y+2)^2}{9} = -1$

$9 + 1 = \frac{(y+2)^2}{9}$

$10 = \frac{(y+2)^2}{9}$

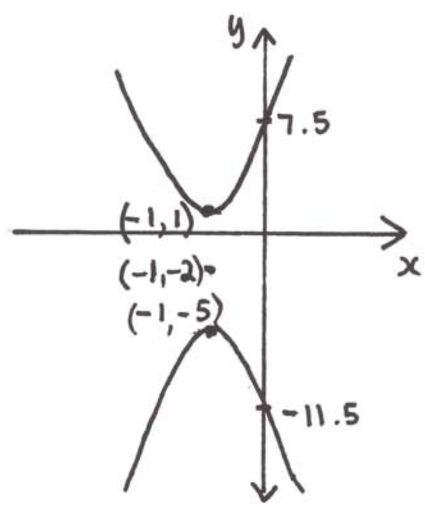
$90 = (y+2)^2$

$y + 2 = \pm\sqrt{90}$

$y = \pm\sqrt{90} - 2$

y-intercepts -11.5 and 7.5

c) Use the information above to sketch the graph of $9(x+1)^2 - \frac{(y+2)^2}{9} = -1$.



Complete Assignment Questions #1 - #9

Assignment

1. Determine the equation of the given hyperbola after the transformation.

a) Determine the equation of the image of the hyperbola $\frac{(x-9)^2}{25} - \frac{y^2}{144} = 1$ after a translation 12 units up and 5 units left.

$x \to x+5$

$y \to y-12$

$$\frac{((x+5)-9)^2}{25} - \frac{(y-12)^2}{144} = 1$$

$$\underline{\underline{\frac{(x-4)^2}{25} - \frac{(y-12)^2}{144} = 1}}$$

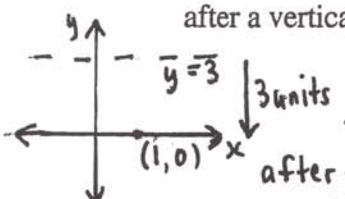
b) Determine the equation of the image of the hyperbola $\frac{x^2}{9} - \frac{(y+2)^2}{16} = 1$ after a horizontal stretch about the line $x = 0$ by a factor 3.

$x \to \frac{1}{3}x$

$$\frac{\left(\frac{1}{3}x\right)^2}{9} - \frac{(y+2)^2}{16} = 1 \qquad \frac{\frac{1}{9}x^2}{9} - \frac{(y+2)^2}{16} = 1$$

$$\underline{\underline{\frac{x^2}{81} - \frac{(y+2)^2}{16} = 1}}$$

c) Determine the equation of the image of the hyperbola $(x-1)^2 - \frac{y^2}{4} = -1$ after a vertical stretch of factor 2 about the line $y = 3$.



centre $(1,0)$ $a = 1$ $b = 2$

the centre $(1,0)$ is 3 units below $y = 3$.
after the stretch it will be $2(3) = 6$ units below
at $(1,-3)$

The vertical by factor 2 changes b from 2
to $2(2) = 4$. a is unchanged.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1 \qquad \underline{\underline{(x-1)^2 - \frac{(y+3)^2}{16} = -1}}$$

or

$y \to \frac{1}{2}y$ followed by

$y \to y + (F-1)L$

$y \to y + (2-1)(3)$ $\quad y \to y+3$

$$(x-1)^2 - \frac{\left(\frac{1}{2}y\right)^2}{4} = -1$$

$$(x-1)^2 - \frac{y^2}{16} = -1$$

$$(x-1)^2 - \frac{(y+3)^2}{16} = -1$$

2. Consider the hyperbola with equation $\frac{(x-5)^2}{64} - \frac{(y+3)^2}{81} = 1$. $\quad a = 8$ $b = 9$ $h = 5$ $k = -3$

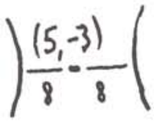
a) Use transformations to describe how the graph of this conic section can be obtained from the graph of the hyperbola $x^2 - y^2 = 1$.

$$\left.\begin{array}{l} x \to \frac{1}{8}x \\ y \to \frac{1}{9}y \end{array}\right\} \quad \left(\tfrac{1}{8}x\right)^2 - \left(\tfrac{1}{9}y^2\right) = 1 \qquad \frac{x^2}{64} - \frac{y^2}{81} = 1$$

$$\left.\begin{array}{l} x \to x-5 \\ y \to y+3 \end{array}\right\} \quad \frac{(x-5)^2}{64} - \frac{(y+3)^2}{81} = 1$$

horizontal stretch by a factor of 8 about the y-axis, vertical stretch by a factor of 9 about the x-axis followed by a translation 5 units right and 3 units down.

b) Determine the following features of the graph of $\frac{(x-5)^2}{64} - \frac{(y+3)^2}{81} = 1$.



i) centre

$(5,-3)$

ii) length of transverse axis

$2 \times 8 = 16$

iii) coordinates of vertices

$(-3,-3)$ and $(13,-3)$

iv) domain

$x \le -3$ or $x \ge 13$

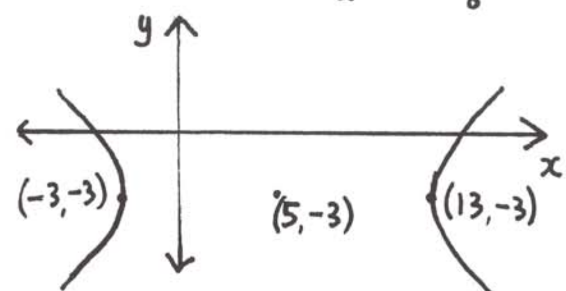
v) range

$y \in R$

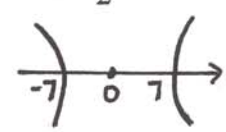
vi) slopes of the asymptotes

$\pm \frac{b}{a} = \pm \frac{9}{8}$

c) Use the information above to sketch the graph of $\frac{(x-5)^2}{64} - \frac{(y+3)^2}{81} = 1$.



3. A hyperbola has vertices at (–7, 0) and (7, 0). One of the asymptotes has a slope $\frac{3}{2}$. Find the equation of the hyperbola in standard form. centre $(0,0)$



$a = 7$ slope $= \frac{3}{2}$

$\frac{b}{a} = \frac{3}{2}$

$b = \frac{3a}{2} = \frac{21}{2}$

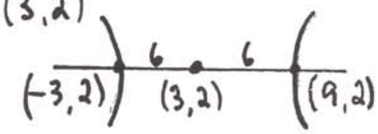
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$\frac{(x-0)^2}{7^2} - \frac{(y-0)^2}{(21/2)^2} = 1$

$\underline{\underline{\frac{x^2}{49} - \frac{4y^2}{441} = 1}}$

4. A hyperbola has vertices at (–3, 2) and (9, 2). The asymptotes have slopes of $\pm\frac{1}{3}$.

a) Find the equation of the hyperbola in standard form. centre $(3,2)$



$a = 6$ slope $= \frac{1}{3}$

$\frac{b}{a} = \frac{1}{3}$

$b = \frac{a}{3} = \frac{6}{3}$

$b = 2$

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$\frac{(x-3)^2}{6^2} - \frac{(y-2)^2}{2^2} = 1$

$\underline{\underline{\frac{(x-3)^2}{36} - \frac{(y-2)^2}{4} = 1}}$

b) The hyperbola is stretched horizontally by a factor of $\frac{2}{3}$ about the line $x = 9$. Determine the equation of the transformed hyperbola.

centre $(3,2)$ $a = 6$, $b = 2$

$(3,2)$

the centre is 6 units left of $x = 9$

$x = 9$

after the *stretch* the centre is $\frac{2}{3}(6) = 4$ units left

6 units

at $(5,2)$

the horizontal *stretch* by factor $\frac{2}{3}$ changes a from 6 to $\frac{2}{3}(6) = 4$. b is unchanged.

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $\underline{\underline{\frac{(x-5)^2}{16} - \frac{(y-2)^2}{4} = 1}}$

or

$x \to \frac{3}{2}x$ followed by

$x \to x + (F-1)L$

$x \to x + (\frac{2}{3}-1)(9)$ $x \to x-3$

$\frac{(\frac{3}{2}x-3)^2}{36} - \frac{(y-2)^2}{4} = 1$

$\frac{(\frac{3}{2}(x-2))^2}{36} - \frac{(y-2)^2}{4} = 1$

$\frac{(x-2)^2}{16} - \frac{(y-2)^2}{4} = 1$ $\frac{((x-3)-2)^2}{16} - \frac{(y-2)^2}{4}$

$\frac{(x-5)^2}{16} - \frac{(y-2)^2}{4} = 1$

5. Consider the hyperbola with equation $16(x+2)^2 - 4(y-6)^2 = -1$.

$\dfrac{(x+2)^2}{1/16} - \dfrac{(y-6)^2}{1/4} = -1$

$a = 1/4 \quad b = 1/2 \quad h = -2 \quad k = 6$

a) Use transformations to describe how the graph of this conic section can be obtained from the graph of the hyperbola $x^2 - y^2 = -1$.

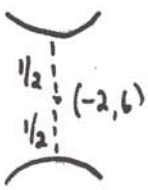
$\left.\begin{array}{l} x \to 4x \\ y \to 2y \end{array}\right\} \quad (4x)^2 - (2y)^2 = -1$

$16x^2 - 4y^2 = -1$

$\left.\begin{array}{l} x \to x+2 \\ y \to y-6 \end{array}\right\} \quad 16(x+2)^2 - 4(y-6)^2 = -1$

horizontal stretch by a factor of $\frac{1}{4}$ about the y-axis, vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, followed by a translation 2 units left and 6 units up

b) Determine the following features of the graph of $16(x+2)^2 - 4(y-6)^2 = -1$.



i) centre

$(-2, 6)$

ii) length of transverse axis

$2 \times \frac{1}{2} = 1$

iii) coordinates of vertices

$\left(-2, \frac{11}{2}\right)$ and $\left(-2, \frac{13}{2}\right)$

iv) domain

$x \in R$

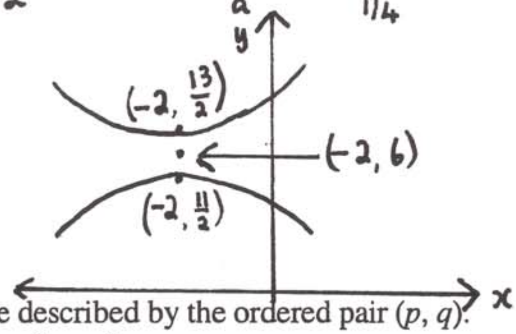
v) range

$y \le \frac{11}{2}$ or $y \ge \frac{13}{2}$

vi) slopes of the asymptotes

$\pm \frac{b}{a} = \pm \frac{1/2}{1/4} = \pm 2$

c) Use the information above to sketch the graph of $16(x+2)^2 - 4(y-6)^2 = -1$.



6. A translation of p units right and q units up can be described by the ordered pair (p, q).

a) Determine the equation of the hyperbola $\dfrac{(y-2)^2}{50} - \dfrac{x^2}{25} = 1$ after a translation described by the ordered pair $(1, -5)$. 1 right, 5 down

$x \to x-1$

$y \to y+5$

$\dfrac{((y+5)-2)^2}{50} - \dfrac{(x-1)^2}{25} = 1$

$\underline{\underline{\dfrac{(y+3)^2}{50} - \dfrac{(x-1)^2}{25} = 1}}$

b) If the point $P(5, 12)$ lies on the original hyperbola, determine the coordinates of P', the image of P, under the transformation in a).

$(5, 12)$ 1 right 5 down $(6, 7)$

c) Determine the slopes of the asymptotes of the hyperbolas in a) and b).

a) $a^2 = 25 \quad b^2 = 50 \quad$ slope $= \pm \frac{b}{a} = \frac{\pm 5\sqrt{2}}{5} = \underline{\pm\sqrt{2}}$

$a = 5 \quad b = \sqrt{50} = 5\sqrt{2}$

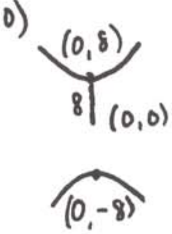
b) Since b) is a translation of a) the slope also $= \underline{\pm\sqrt{2}}$

Multiple Choice

7. A hyperbola has asymptotes with slopes $\pm\frac{4}{3}$. If the vertices are (0, –8) and (0, 8) the equation of the hyperbola is

centre (0, 0)

$b = 8$



A. $\frac{x^2}{36} - \frac{y^2}{64} = 1$

B. $\frac{x^2}{36} - \frac{y^2}{64} = -1$

C. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

D. $\frac{x^2}{9} - \frac{y^2}{16} = -1$

$\frac{b}{a} = \frac{4}{3}$

$\frac{8}{a} = \frac{4}{3}$

$4a = 24$

$a = 6$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\frac{x^2}{36} - \frac{y^2}{64} = -1$

8. The domain of the quadratic relation $\frac{(y-8)^2}{64} - \frac{(x-2)^2}{4} = 1$ is

A. $x \leq -2$ or $x \geq 6$

B. $x \leq 0$ or $x \geq 4$

C. $x \leq -4$ or $x \geq 0$

D. $x \in \Re$

$\frac{(x-2)^2}{4} - \frac{(y-8)^2}{64} = -1$

opens along a vertical axis

Numerical Response

9. The slopes of the asymptotes of the hyperbola $\frac{4(x+6)^2}{9} - \frac{(y-1)^2}{9} = 1$ are $\pm k$, where $k > 0$. The value of k, to the nearest tenth, is _____.

(Record your answer in the numerical response box from left to right)

2	.	0	

$a^2 = \frac{9}{4}$ $\quad b^2 = 9$ $\quad \frac{b}{a} = \frac{3}{3/2} = 2$

$a = \frac{3}{2}$ $\quad b = 3$

slopes are ± 2 $\quad k = 2$

Answer Key

1. a) $\dfrac{(x-4)^2}{25} - \dfrac{(y-12)^2}{144} = 1$ **b)** $\dfrac{x^2}{81} - \dfrac{(y+2)^2}{16} = 1$ **c)** $(x-1)^2 - \dfrac{(y+3)^2}{16} = -1$

2. a) horizontal stretch by a factor of 8 about the y-axis, a vertical stretch by a factor of 9 about the x-axis, followed by a translation 5 units right and 3 units down.

b) i) $(5, -3)$ **ii)** 16 **iii)** $(-3, -3)$ and $(13, -3)$

iv) $x \leq -3$ or $x \geq 13$ **v)** $y \in \Re$ **vi)** $\pm\frac{9}{8}$

3. $\dfrac{x^2}{49} - \dfrac{4y^2}{441} = 1$

4. a) $\dfrac{(x-3)^2}{36} - \dfrac{(y-2)^2}{4} = 1$ **b)** $\dfrac{(x-5)^2}{16} - \dfrac{(y-2)^2}{4} = 1$

5. a) horizontal stretch by a factor of $\dfrac{1}{4}$ about the y-axis, a vertical stretch by a factor of $\dfrac{1}{2}$ about the x-axis, followed by a translation 2 units left and 6 units up.

b) i) $(-2, 6)$ **ii)** 1 **iii)** $\left(-2, \frac{11}{2}\right)$ and $\left(-2, \frac{13}{2}\right)$

iv) $x \in \Re$ **v)** $y \leq \frac{11}{2}$ or $y \geq \frac{13}{2}$ **vi)** ± 2

6. a) $\dfrac{(y+3)^2}{50} - \dfrac{(x-1)^2}{25} = 1$ **b)** $(6, 7)$ **c)** $\pm\sqrt{2}$

7. B **8.** D **9.**

2	.	0	

Conic Sections Lesson #7: Applications of Conic Sections

Sometimes mathematicians have a habit of studying topics to keep their skills sharp or just for fun. At the time, some of these topics may appear to have little practical us, but then many years or even centuries later, these topics turn out to have great scientific value.

Appollonius' study of conic sections is such a topic. His work with conic sections a large number of applications in our current society.

- Bodies projected upward and obliquely to the pull of gravity in nature (such as the path of a golf ball after it has been struck by a golf club, the design of a headlight of a car or enlarger bulb) and the design of parabolic mirrors in telescopes may be approximated by a parabola. The largest parabolic mirror used in a telescope (approximately 6 m in diameter) is located in the Caucasus mountains of Russia and was built in 1967. However, a company, UPC, is currently building a telescope with a parabolic mirror of 10 m in length in Europe.

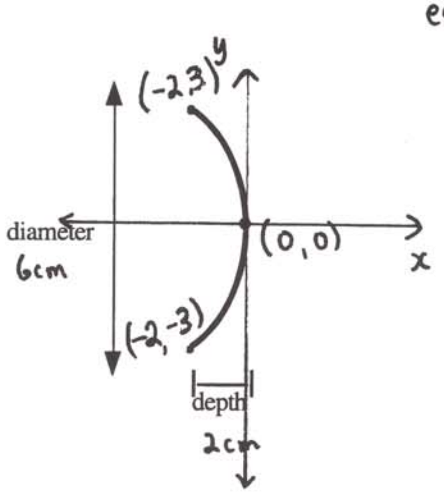
- The path of Halley's Comet, the light path of lithotripsy (a medical procedure for treating kidney stones), and many building designs all follow the path of an ellipse.

- Planes or ships at sea may use LORAN, a navigation system which uses electronic signals in a hyperbolic path to determine the location of a ship or plane. Circular cones intersected by a plane parallel to the axis such as sharpening a pencil or a sonic boom shock wave from a jet follow part of the path of a hyperbola. A hyperbolic path is also used in building designs such as the Saddledome in Calgary.

A special enlarger bulb is designed to enlarge photographs from a 4 x 5 enlarger so that the reflector takes the shape of a parabola if viewed from its side. The diameter of the reflector is 6 cm and the depth of the reflector is 2 cm.

Find the equation of the parabola in standard form if the vertex of the parabola is located at (0, 0) and it opens to the left.

equation is of the form $x - h = a(y-k)^2$



vertex is $(0,0)$ so $h=0$, $k=0$

$x = ay^2$

The point $(-2, 3)$ lies on the parabola

$-2 = a(3^2)$

$-2 = 9a$

$a = \frac{-2}{9}$

equation is $\underline{\underline{x = -\frac{2}{9}y^2}}$

For security reasons, a military plane is designated to fly curved air routes to transport top secret material across the ocean between islands. During one of these flights a pilot is instructed to fly part of a hyperbolic path between two islands passing over an oil rig equidistant from both islands.

If the domain of the flight path is $\{x \mid 0 \le x \le 40\}$ and the range is $\{y \mid -150 \le y \le 150\}$ determine the equation of the hyperbolic path in standard form, where $a = 10$.

domain $0 \le x \le 40$
range $-150 \le y \le 150$ places the origin at the oil rig.

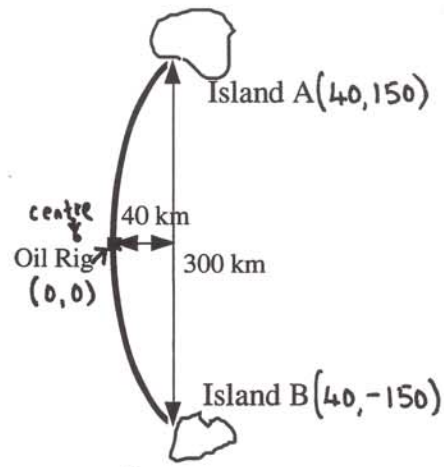
hyperbola is of the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

If $a = 10$, the centre of the hyperbola is 10 km left of the oil rig at $(-10, 0)$ so $h = -10$, $k = 0$

$\dfrac{(x+10)^2}{100} - \dfrac{y^2}{b^2} = 1$. replace $(40, 150)$ into the equation.

$\dfrac{(40+10)^2}{100} - \dfrac{150^2}{b^2} = 1$, $25 - \dfrac{22500}{b^2} = 1$

$24 = \dfrac{22500}{b^2}$, $24b^2 = 22500$ $b^2 = \dfrac{1875}{2}$



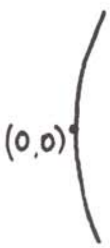
Equation $\dfrac{(x+10)^2}{100} - \dfrac{2y^2}{1875} = 1$

Complete Assignment Questions #1 - #5

Assignment

1. Use the information from Class Ex. #2 to answer the following.

   a) As an alternate route, a pilot is instructed to fly a parabolic path between the islands passing over the oil rig. Determine the equation of the parabola in standard form.



vertex of parabola is $(0,0)$ equation is of the form $x - h = a(y-k)^2$

$h = 0$ $k = 0$

$x = ay^2$

replace $(40, 150)$

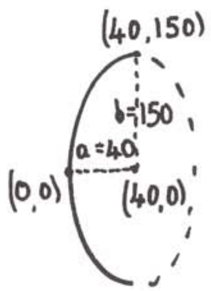
$40 = a(150)^2$

$40 = 22500a$

$a = \dfrac{2}{1125}$

equation $x = \dfrac{2}{1125}y^2$

   b) As a third alternate route a pilot has been instructed to fly a semi-elliptical path with centre (40, 0) between the islands passing over the oil rig. Determine the equation of the semi-ellipse in standard form.



centre $(40, 0)$ from the diagram equation is of the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

$h = 40$ $k = 0$ $a = 40$ $b = 150$

$\dfrac{(x-40)^2}{40^2} + \dfrac{(y-0)^2}{150^2} = 1$

$\dfrac{(x-40)^2}{1600} + \dfrac{y^2}{22500} = 1$

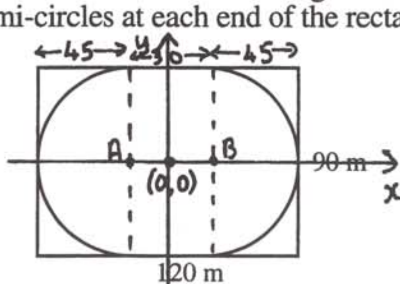
c) The coordinate system is changed so that the centre of the ellipse in b) is (0, 0). How can we use the answer in b) to determine the new equation of the semi-ellipse? Determine the new equation of the semi-ellipse in standard form.

The centre has moved from (40,0) to (0,0) a translation 40 km left.

Replacing x by x+40 in the answer to b) will give the new equation.

$$\frac{(x+40-40)^2}{1600}+\frac{y^2}{22500}=1 \qquad \frac{x^2}{1600}+\frac{y^2}{22500}=1$$

2. A competitive ice skating facility is to be designed for the upcoming championships within a 120 m x 90 m wide rectangular area. The design committee is considering a design with semi-circles at each end of the rectangle.



semicircles have a diameter of 90 m so the radius is 45 m.

$AB = 120 - 2(45) = 30$ m

so $A(-15, 0)$ $B(15, 0)$

a) Determine the equation, in general form, of both semi-circles using the origin at the centre of the rectangle. State the domain and range for each semi-circle.

left semi-circle

centre $(-15, 0)$ radius $= 45$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{a^2}=1$$

$$\frac{(x+15)^2}{45^2}+\frac{y^2}{45^2}=1$$

$$(x+15)^2+y^2=45^2$$

$$x^2+30x+225+y^2=2025$$

$$x^2+y^2+30x-1800=0$$

domain $\{x \mid -60 \le x \le -15, x \in R\}$

range $\{y \mid -45 \le y \le 45, y \in R\}$

right semi-circle

centre $(15, 0)$ radius 45

$$\frac{(x-15)^2}{45^2}+\frac{y^2}{45^2}=1$$

$$(x-15)^2+y^2=45^2$$

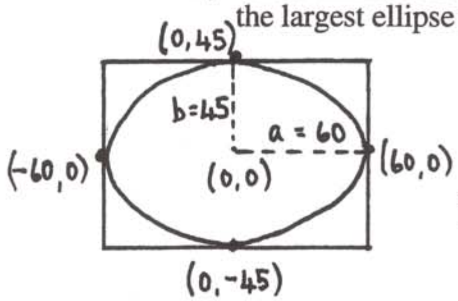
$$x^2-30x+225+y^2=2025$$

$$x^2+y^2-30x-1800=0$$

domain: $\{x \mid 15 \le x \le 60, x \in R\}$

range: $\{y \mid -45 \le y \le 45, y \in R\}$

b) The committee is also considering an elliptical ice surface. Determine the equation of the largest ellipse possible.



$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

centre $(0,0)$ $h=0$ $k=0$

From the diagram $a=60$, $b=45$.

$$\frac{x^2}{60^2}+\frac{y^2}{45^2}=1$$

$$\frac{x^2}{3600}+\frac{y^2}{2025}=1$$

3. A bridge with a curved arch support is to be constructed over a small river. The curved arch support of the bridge is to be 10 metres high and 16 metres wide.

a) If the origin of the coordinate system is taken at the extreme left edge of the curved support, determine the equation of the curve in standard form if it is to be constructed in the form of;

i) a semi-ellipse

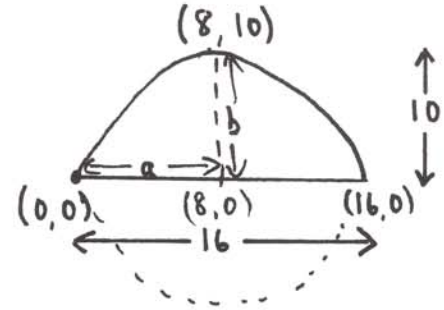
centre $(8,0)$

$a = 8 \quad b = 10$

equation $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

$$\frac{(x-8)^2}{8^2} + \frac{y^2}{10^2} = 1$$

equation is $\underline{\underline{\dfrac{(x-8)^2}{64} + \dfrac{y^2}{100} = 1}}$



ii) a parabola.

vertex $(8,10)$

equation $y - k = a(x-h)^2$

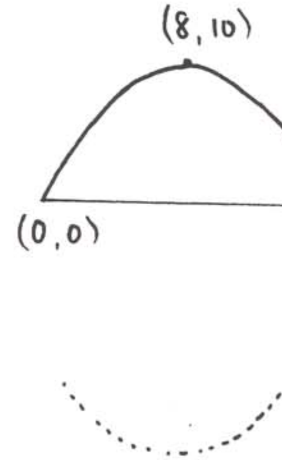
$y - 10 = a(x-8)^2$

replace $(0,0)$ $\quad 0 - 10 = a(0-8)^2$

$-10 = a(64)$

$a = \dfrac{-5}{32}$

equation is $\underline{\underline{y - 10 = \dfrac{-5}{32}(x-8)^2}}$



iii) the lower branch of a hyperbola where $b = 4$.

vertex $(8,10)$ with $b = 4$

so centre $(8,14)$

equation $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = -1$

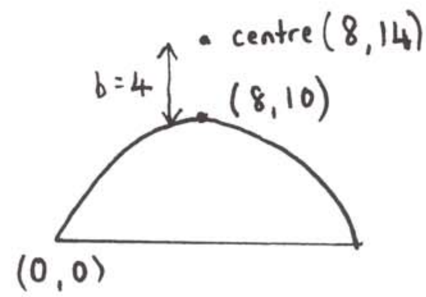
$$\frac{(x-8)^2}{a^2} - \frac{(y-14)^2}{4^2} = -1$$

replace $(0,0)$ $\quad \dfrac{(0-8)^2}{a^2} - \dfrac{(0-14)^2}{16} = -1$

$\dfrac{64}{a^2} - \dfrac{196}{16} = -1$

$\dfrac{64}{a^2} = \dfrac{45}{4} \qquad 45a^2 = 256$

$a^2 = \dfrac{256}{45}$



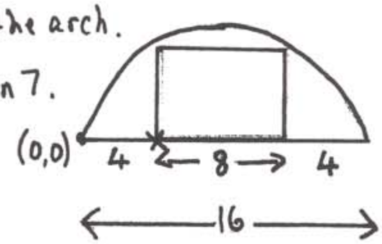
equation is $\dfrac{(x-8)^2}{256/45} - \dfrac{(y-14)^2}{16} = -1$

$$\underline{\underline{\frac{45(x-8)^2}{256} - \frac{(y-14)^2}{16} = -1}}$$

b) A loaded rectangular barge 7 metres high above the water and 8 metres wide will be travelling under the bridge after it is constructed. Determine which of the three curved arch supports in a) will allow the barge to pass safely under the bridge. Calculate the clearance in each case to the nearest tenth of a metre.

Assume the barge passes under the centre of the arch.

At $x = 4$ the value of y must be greater than 7.



semi-ellipse

$$\frac{(x-8)^2}{64} + \frac{y^2}{100} = 1$$

At $x = 4$, $\frac{(4-8)^2}{64} + \frac{y^2}{100} = 1$

$$\frac{1}{4} + \frac{y^2}{100} = 1$$

$$\frac{y^2}{100} = \frac{3}{4}$$

$$y^2 = \frac{300}{4}$$

$$y = \pm\sqrt{\frac{300}{4}}$$

for semi-ellipse $y = \sqrt{\frac{300}{4}} = \sqrt{75} = 8.66$

Semi-ellipse will allow the barge to pass safely. Clearance $= 8.66 - 7$

$= 1.7$ m (nearest tenth)

parabola

$$y - 10 = -\frac{5}{32}(x-8)^2$$

At $x = 4$, $y - 10 = -\frac{5}{32}(4-8)^2$

$$y - 10 = -\frac{5}{2}$$

$$y = 7.5$$

Parabola will allow the barge to pass safely.

Clearance $= 7.5 - 7 = 0.5$ m

hyperbola

$$\frac{45(x-8)^2}{256} - \frac{(y-14)^2}{16} = -1$$

At $x = 4$ $\frac{45(4-8)^2}{256} - \frac{(y-14)^2}{16} = -1$

$$\frac{45}{16} - \frac{(y-14)^2}{16} = -1$$

$$45 - (y-14)^2 = -16$$

$$61 = (y-14)^2$$

$$y - 14 = \pm\sqrt{61}$$

$$y = 14 \pm \sqrt{61}$$

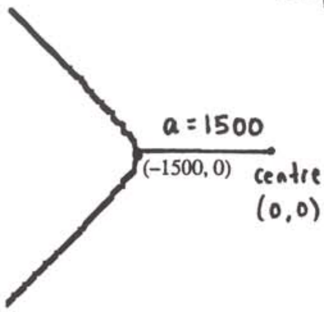
for lower branch $y = 14 - \sqrt{61} = 6.2$ (nearest tenth)

Hyperbola will not allow the barge to pass safely.

Fails by 0.8 m

4. When the XL-17 jet breaks the sound barrier, the shock wave that is produced at the surface of the earth is hyperbolic in shape.

Determine the equation of the hyperbola in standard form if the slope of one of the asymptotes of the hyperbola is $-\frac{3}{2}$, the centre is located at (0, 0) and the vertex is located at (–1500, 0).



slope $= -\frac{3}{2} = -\frac{b}{a}$

$-3a = -2b$

$-3(1500) = -2b$

$-4500 = -2b$

$b = 2250$

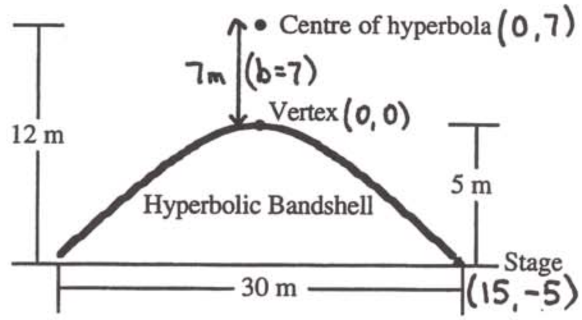
equation is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ (h. k.)

$\frac{x^2}{1500^2} - \frac{y^2}{2250^2} = 1$

$$\frac{x^2}{2250000} - \frac{y^2}{5062500} = 1$$

5. The Musical Arts Entertainment company is constructing a bandshell to obtain a high quality sound for it musical shows. The bandshell is to be constructed in a hyperbolic shape as shown.

Using the vertex as the origin determine the equation of the hyperbola in standard form. State the domain and range.



equation is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$

centre (0,7) $h=0$ $k=7$ $b=7$

$\frac{x^2}{a^2} - \frac{(y-7)^2}{7^2} = -1$

replace (15,-5) $\frac{15^2}{a^2} - \frac{(-5-7)^2}{49} = -1$

$\frac{225}{a^2} - \frac{144}{49} = -1$

$\frac{225}{a^2} = \frac{95}{49}$

$95a^2 = 11025$, $a^2 = \frac{11025}{95} = \frac{2205}{19}$

equation is $\frac{19x^2}{2205} - \frac{(y-7)^2}{49} = -1$

domain $\{x \mid -15 \le x \le 15, x \in R\}$

range $\{y \mid -5 \le y \le 0, y \in R\}$

Answer Key

1. a) $x = \frac{2}{1125}y^2$ **b)** $\frac{(x-40)^2}{1600} + \frac{y^2}{22500} = 1$ **c)** $\frac{x^2}{1600} + \frac{y^2}{22500} = 1$

2. a) Left semi-circle $x^2 + y^2 + 30x - 1800 = 0$
Domain: $\{x \mid -60 \le x \le -15, x \in \Re\}$
Range: $\{y \mid -45 \le y \le 45, y \in \Re\}$

Right semi-circle $x^2 + y^2 - 30x - 1800 = 0$
Domain: $\{x \mid 15 \le x \le 60, x \in \Re\}$
Range: $\{y \mid -45 \le y \le 45, y \in \Re\}$

b) $\frac{x^2}{3600} + \frac{y^2}{2025} = 1$

3. a) i) $\frac{(x-8)^2}{64} + \frac{y^2}{100} = 1$ **ii)** $y - 10 = -\frac{5}{32}(x-8)^2$ **iii)** $\frac{45(x-8)^2}{256} - \frac{(y-14)^2}{16} = -1$

b) Ellipse: yes by 1.7 m Parabola: yes by 0.5 m Hyperbola: No by 0.8 m

4. $\frac{x^2}{2\,250\,000} - \frac{y^2}{5\,062\,500} = 1$

5. $\frac{19x^2}{2205} - \frac{(y-7)^2}{49} = -1$ Domain: $\{x \mid -15 \le x \le -15, x \in \Re\}$
Range: $\{y \mid -5 \le y \le 0, y \in \Re\}$

Conics Lesson #8:
Converting From General To Standard Form

Warm-Up #1

Quadratic relations, or conics, can be written in two forms:

- general form: $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A, C, D, E, F, \in I$.

  or

- standard form: $\dfrac{(x-h)^2}{a^2} \pm \dfrac{(y-k)^2}{b^2} = 1$

  $y - k = a(x-h)^2$

  $x - h = a(y-k)^2$

In order to convert equations of conics sections from general to standard form we will apply the method of **completing the square** learned in earlier math courses.

Completing the Square

Recall the instructions shown used for the method of **completing the square**.

Step 1		Step 2		Step 3		Step 4
Factor the leading coefficient out of the x^2 term and the x term.	→	Take half of the numerical coefficient of the x-term and square it.	→	Add the squared number in step 2 to both sides of the equation.	→	Simplify and write in factored form.

Consider the equation $x^2 + 10x - y + 16 = 0$.

a) Convert $x^2 + 10x - y + 16 = 0$ in standard form.

$x^2 + 10x + 25 - y + 16 = 0 + 25$

$(x+5)^2 = y - 16 + 25$

$\underline{\underline{y + 9 = (x+5)^2}}$

b) Verify the answer in a) use a graphing calculator to graph both equations.

c) Use the standard form to determine the domain, range, and vertex of the parabola.

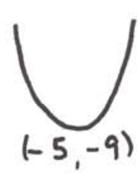
$y - k = a(x-h)^2$

$y + 9 = (x+5)^2$

vertex $(-5, -9)$

domain $x \in R$

range $\{y \mid y \geq -9, y \in R\}$



Class Ex. #2

Consider the quadratic relation with equation $x^2 - 3y^2 - 10x - 24y - 59 = 0$.
Convert the equation to standard form and determine the vertices of the conic section.

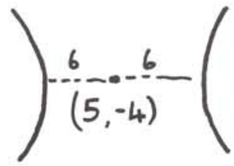
$x^2 - 10x \qquad - 3(y^2 + 8y \qquad) = 59$

$x^2 - 10x + 25 \quad - 3(y^2 + 8y + 16) = 59 + 25 - 3(16)$

$(x-5)^2 \quad - 3(y+4)^2 = 36$

$\dfrac{(x-5)^2}{36} - \dfrac{(y+4)^2}{12} = 1$

hyperbola



centre $(5,-4)$ $\quad a^2 = 36 \quad a = 6$

vertices $(-1,-4)$ and $(11,-4)$

Class Ex. #3

The equation $9x^2 + 25y^2 - 108x + 150y + 324 = 0$ represents an ellipse.

a) List the following general characteristics of the graph of the ellipse
- centre
- domain and range
- vertices
- x- and y- intercepts

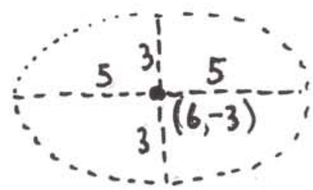
$9(x^2 - 12x \qquad) + 25(y^2 + 6y \qquad) = -324$

$9(x^2 - 12x + 36) + 25(y^2 + 6y + 9) = -324 + 9(36) + 25(9)$

$9(x-6)^2 + 25(y+3)^2 = 225$

$\dfrac{(x-6)^2}{25} + \dfrac{(y+3)^2}{9} = 1$

ellipse : $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ $\quad a = 5 \quad h = 6$
$\quad b = 3 \quad k = -3$



centre $(6,-3)$
vertices $(1,-3)$ and $(11,-3)$
domain $\{x \mid 1 \le x \le 11,\ x \in R\}$
range $\{y \mid -6 \le y \le 0,\ y \in R\}$
x-intercept 6
no y-intercept

x-int. $y = 0$

$9x^2 - 108x + 324 = 0$

$9(x^2 - 12x + 36) = 0$

$9(x-6)^2 = 0$

$x = 6$

y-int. $x = 0$

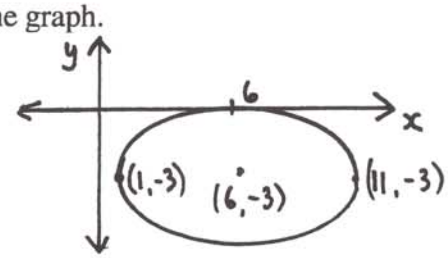
$25y^2 + 150y + 324 = 0$

quadra

$y = \dfrac{-150 \pm \sqrt{150^2 - 32400}}{50}$

no real roots

b) Use these results to sketch the graph.



The equation $2x^2 + 2y^2 + 16x - 5y + 8 = 0$ represents a circle.

a) Convert the equation to the form $(x - h)^2 + (y - k)^2 = r^2$.

$$2(x^2 + 8x \quad) + 2(y^2 - \tfrac{5}{2}y \quad) = -8$$
$$2(x^2 + 8x + 16) + 2(y^2 - \tfrac{5}{2}y + \tfrac{25}{16}) = -8 + 2(16) + 2\left(\tfrac{25}{16}\right)$$
$$2(x + 4)^2 + 2(y - \tfrac{5}{4})^2 = \frac{217}{8}$$
$$(x + 4)^2 + (y - \tfrac{5}{4})^2 = \frac{217}{16}$$

b) Determine the centre and radius (to the nearest tenth).

centre $\left(-4, \frac{5}{4}\right)$ $\quad r^2 = \frac{217}{16}$ $\quad r = \sqrt{\frac{217}{16}}$ $\quad$ radius = 3.7 (nearest tenth)

Complete Assignment Questions #1 - #9

Assignment

1. Convert the following equations to standard form and determine the type of conic that each represents.

a) $x^2 - 6x - y - 10 = 0$

$$x^2 - 6x + 9 - y - 10 = 0 + 9$$
$$(x - 3)^2 = y + 10 + 9$$
$$y + 19 = (x - 3)^2 \qquad \text{parabola}$$

b) $x^2 + 3y^2 + 10x - 30y + 91 = 0$

$$x^2 + 10x \quad + 3(y^2 - 10y \quad) = -91$$
$$x^2 + 10x + 25 + 3(y^2 - 10y + 25) = -91 + 25 + 3(25)$$
$$(x + 5)^2 + 3(y - 5)^2 = 9$$
$$\frac{(x + 5)^2}{9} + \frac{(y - 5)^2}{3} = 1 \qquad \text{ellipse}$$

c) $16x^2 - y^2 - 96x + 8y + 112 = 0$

$16(x^2 - 6x \quad) - (y^2 - 8y \quad) = -112$

$16(x^2 - 6x + 9) - (y^2 - 8y + 16) = -112 + 16(9) - 16$

$16(x-3)^2 - (y-4)^2 = 16$

$(x-3)^2 - \frac{(y-4)^2}{16} = 1$ hyperbola

d) $-2y^2 - x + 20y - 47 = 0$

$-2(y^2 - 10y \quad) = x + 47$

$-2(y^2 - 10y + 25) = x + 47 - 2(25)$

$-2(y-5)^2 = x - 3$

$x - 3 = -2(y-5)^2$ parabola

e) $4x^2 - y^2 - 24x + 52 = 0$

$4(x^2 - 6x \quad) - y^2 = -52$

$4(x^2 - 6x + 9) - y^2 = -52 + 4(9)$

$4(x-3)^2 - y^2 = -16$

$\frac{(x-3)^2}{4} - \frac{y^2}{16} = -1$ hyperbola

2. Find the centre and radius of each circle.

a) $x^2 + y^2 - 8x - 6y + 9 = 0$

$x^2 - 8x \quad + y^2 - 6y \quad = -9$

$x^2 - 8x + 16 + y^2 - 6y + 9 = -9 + 16 + 9$

$(x-4)^2 + (y-3)^2 = 16$

Centre $(4, 3)$

radius $= 4$

b) $x^2 + y^2 - 4x + 3y = 0$

$x^2 - 4x \quad + y^2 + 3y \quad = 0$

$x^2 - 4x + 4 + y^2 + 3y + \frac{9}{4} = \frac{9}{4}$

$(x-2)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{9}{4}$

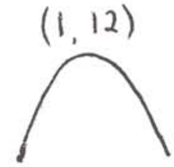
Centre $\left(2, -\frac{3}{2}\right)$

radius $= \frac{3}{2}$

3. Consider the conic section with equation $3x^2 - 6x + y - 9 = 0$.

a) Convert $3x^2 - 6x + y - 9 = 0$ to standard form.

$$3(x^2 - 2x \quad) = -y + 9$$
$$3(x^2 - 2x + 1) = -y + 9 + 3(1)$$
$$3(x-1)^2 = -y + 12$$
$$y - 12 = -3(x-1)^2$$



b) Determine the domain, range, and vertex of the graph of the conic section.

domain $x \in R$ range $y \leq 12$ vertex $(1, 12)$

c) Find the x-intercepts and y-intercepts and sketch the graph.

$\underline{y = 0}$

$-3(x-1)^2 = -12$

$(x-1)^2 = 4$

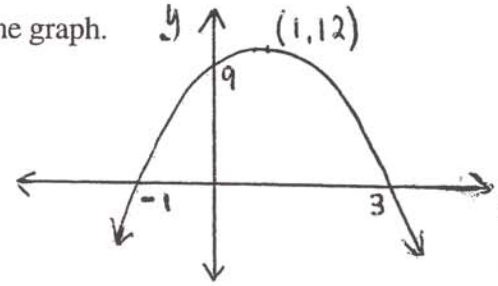
$x - 1 = \pm 2$

x intercepts: -1 and 3

$\underline{x = 0}$

$y - 12 = -3(1)$

y intercept : 9



4. Consider the equation $16x^2 + 9y^2 + 192x - 36y + 468 = 0$.

a) Describe the series of transformations applied to the graph of the unit circle $x^2 + y^2 = 1$ which would result in the graph of the equation $16x^2 + 9y^2 + 192x - 36y + 468 = 0$.

$$16(x^2 + 12x \quad) + 9(y^2 - 4y \quad) = -468$$
$$16(x^2 + 12x + 36) + 9(y^2 - 4y + 4) = -468 + 16(36) + 9(4)$$
$$16(x+6)^2 + 9(y-2)^2 = 144$$
$$\frac{(x+6)^2}{9} + \frac{(y-2)^2}{16} = 1$$

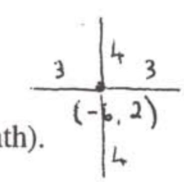
$$\begin{cases} x \to \frac{1}{3}x \\ y \to \frac{1}{4}y \end{cases} \quad \frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$\begin{cases} x \to x+6 \\ y \to y-2 \end{cases} \quad \frac{(x+6)^2}{9} + \frac{(y-2)^2}{16} = 1$$

horizontal stretch by a factor of 3 and a vertical stretch by a factor of 4 followed by a translation 6 units left and 2 units up.

b) Determine the following general characteristics of the graph and sketch the graph:

- centre
- vertices
- domain and range
- the length of the horizontal diameter
- x- and y- intercepts (to the nearest tenth).



centre $(-6, 2)$ vertices $(-6, -2)$ and $(-6, 6)$

domain $-9 \leq x \leq -3$ range $-2 \leq y \leq 6$

length of horizontal diameter $= 2 \times 3 = 6$

$\underline{x = 0}$

$$\frac{36}{9} + \frac{(y-2)^2}{16} = 1$$
$$\frac{(y-2)^2}{16} = -3$$

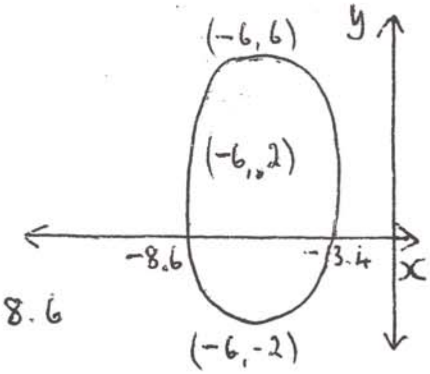
no y-intercept

$\underline{y = 0}$

$$\frac{(x+6)^2}{9} + \frac{4}{16} = 1$$
$$\frac{(x+6)^2}{9} = \frac{3}{4}$$
$$(x+6)^2 = \frac{27}{4}$$

$$x + 6 = \frac{\pm\sqrt{27}}{2}$$

x intercept $-3.4, -8.6$



Multiple Choice

5. The centre of the circle with equation $x^2 + y^2 + 2x - 2y - 25 = 0$ is

   A. $(-2, 2)$ **B.** $(2, -2)$ **C.** $(-1, 1)$ **D.** $(1, -1)$

$x^2 + 2x + 1 + y^2 - 2y + 1 = 25 + 1 + 1$

$(x+1)^2 + (y-1)^2 = 27$ centre$(-1, 1)$

6. The vertex of the parabola with equation $y^2 - 8x - 6y - 7 = 0$ is

   A. $(-2, 3)$
   B. $(3, -2)$
   C. $(-4, 3)$
   D. $(3, -4)$

$y^2 - 6y + 9 = 8x + 7 + 9$

$(y-3)^2 = 8x + 16$

$(y-3)^2 = 8(x+2)$ centre $(-2, 3)$

$(y-k)^2 = a(x-h)$

7. The centre of the ellipse with equation $4x^2 + y^2 - 8x + 4y - 8 = 0$ is

   A. $(-4, 2)$
   B. $(4, -2)$
   C. $(2, -1)$
   D. $(1, -2)$

$4(x^2 - 2x \quad) + y^2 + 4y = 8$

$4(x^2 - 2x + 1) + y^2 + 4y + 4 = 8 + 4(1) + 4$

$4(x-1)^2 + (y+2)^2 = 16$ $\quad \dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{16} = 1$ centre$(1, -2)$

Numerical Response

8. The equation $x^2 - y^2 - 4x + 8y - 21 = 0$ can be written in the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

   The value of $h + k$, to the nearest tenth, is _____ .

   (Record your answer in the numerical response box from left to right)

6	.	0	

$x^2 - 4x + 4 - (y^2 - 8y + 16) = 21 + 4 - 16$

$(x-2)^2 - (y-4)^2 = 9$ $\quad \dfrac{(x-2)^2}{9} - \dfrac{(y-4)^2}{9} = 1$ $\quad h = 2 \quad k = 4 \quad h + k = 6$

9. The circle with equation $x^2 + y^2 - 5x - 7 = 0$ has a radius of k units. The value of k, to the nearest hundredth, is _____ .

   (Record your answer in the numerical response box from left to right)

3	.	6	4

$x^2 - 5x + \frac{25}{4} + y^2 = 7 + \frac{25}{4}$ radius: $\sqrt{\frac{53}{4}}$

$(x - \frac{5}{2})^2 + y^2 = \frac{53}{4}$ $\quad k = 3.64$

Answer Key

1. a) $y + 19 = (x-3)^2$ parabola **b)** $\dfrac{(x+5)^2}{9} + \dfrac{(y-5)^2}{3} = 1$ ellipse

c) $(x-3)^2 - \dfrac{(y-4)^2}{16} = 1$ hyperbola **d)** $x - 3 = -2(y-5)^2$ parabola

e) $\dfrac{(x-3)^2}{4} - \dfrac{y^2}{16} = -1$ hyperbola **2. a)** centre $(4, 3)$ radius 4 **b)** centre $(2, -\frac{3}{2})$ radius $\frac{5}{2}$

3. a) $y - 12 = -3(x-1)^2$ **b)** domain: $x \in \Re$ range: $\{y \mid y \le 12, y \in \Re\}$ vertex $(1, 12)$

c) x–intercepts are -1 and 3 y-intercept $= 9$

4. a) horizontal stretch by a factor of 3 about the y-axis and a vertical stretch by a factor of 4 about the x-axis, followed by a translation 6 units left and two units up.

b) centre $(-6, 2)$ vertices $(-6, -2)$ and $(-6, 6)$ domain: $\{x \mid -9 \le x \le -3, x \in \Re\}$ range: $\{y \mid -2 \le y \le 6, y \in \Re\}$ horizontal diameter = 6 x–intercepts are -8.6 and -3.4 no y-intercept

5. C **6.** A **7.** D **8.**

6	.	0	

9.

3	.	6	4

Conic Sections Lesson #9:
Degenerate Conic Sections and Summary

Degenerate Conic Sections

Consider the equation $4x^2 + 4y^2 + 28x - 20y + 74 = 0$.

a) Which conic section is suggested by the equation? circle

b) Rewrite the equation in standard form.

$$4(x^2+7x \quad) + 4(y^2-5y \quad) = -74$$
$$4\left(x^2+7x+\tfrac{49}{4}\right) + 4\left(y^2-5y+\tfrac{25}{4}\right) = -74 + 4\left(\tfrac{49}{4}\right) + 4\left(\tfrac{25}{4}\right)$$
$$4\left(x+\tfrac{7}{2}\right)^2 + 4\left(y-\tfrac{5}{2}\right)^2 = 0$$
$$\left(x+\tfrac{7}{2}\right)^2 + \left(y-\tfrac{5}{2}\right)^2 = 0$$

c) In what way is the equation in b) different from the usual form for a circle.

right side = 0 not 1 (i.e radius = 0)

d) A circle degenerates to a point when the radius is equal to zero.
Solve the equation in b) to find the coordinates of the point represented by the equation.

$\left(-\tfrac{7}{2}, \tfrac{5}{2}\right)$

Consider the equation $9x^2 - 16y^2 + 72x - 64y + 80 = 0$.

a) Which conic section is suggested by the equation? hyperbola

b) Rewrite the equation in standard form.

$$9(x^2+8x \quad) - 16(y^2+4y \quad) = -80$$
$$9(x^2+8x+16) - 16(y^2+4y+4) = -80 + 9(16) - 16(4)$$
$$9(x+4)^2 - 16(y+2)^2 = 0$$

c) In what way is the equation in b) different from the usual form for a hyperbola.

right side = 0

d) The degenerate of a hyperbola is two intersecting lines.
Solve the equation in b) using a difference of squares to determine the equations of the two intersecting lines.

$$9(x+4)^2 = 16(y+2)^2$$
$$3(x+4) = \pm 4(y+2)$$
$$3x+12 = \pm(4y+8)$$

$3x+12 = -4y-8$

$\underline{\underline{3x+4y+20=0}}$

$3x+12 = 4y+8$

$\underline{\underline{3x-4y+4=0}}$

Equations of Degenerate Conic Sections

Circle

Equation	Primary or Degenerate Conic	Graph
$x^2 + y^2 = 1$	circle	circle
$x^2 + y^2 = 0$	degenerate of a circle	point
$x^2 + y^2 = -1$	degenerate of a circle	no graph

Ellipse

Equation	Primary or Degenerate Conic	Graph
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	ellipse	ellipse
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	degenerate of an ellipse	point
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$	degenerate of an ellipse	no graph

Hyperbola

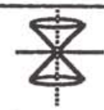
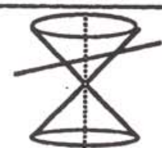
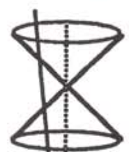
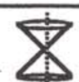
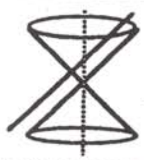
Equation	Primary or Degenerate Conic	Graph
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$	hyperbola	hyperbola
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	degenerate of a hyperbola	two intersecting lines

Parabola

Equation	Primary or Degenerate Conic	Graph
$Ax^2 + Cy^2 + Dx + Ey + F = 0$ with A or C (not both) $= 0$	parabola	parabola
$Ax^2 + Cy^2 + Dx + Ey + F = 0$ with $A = C = 0$	degenerate parabola	one straight line
$Ax^2 + Dx + F = 0$ or $Cy^2 + Ey + F = 0$	degenerate parabola	two parallel lines or one straight line or no graph

Complete Assignment Questions #1 - #5

Summary of Conics

CONIC	Generation of Conic Section	Degenerate Conic	General Form $Ax^2 + Cy^2 + Dx + Ey + F = 0, \quad A, C, D, E, F, \in I$	Standard Form
Circle	 Conic is cut at an angle perpendicular to the central axis	 point or no graph	If $A = C$, then the conic is a circle.	$(x-h)^2 + (y-k)^2 = r^2$ or $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1,$ • centre (h, k)
Ellipse	 Conic is cut at an angle greater than the generator angle	 point or no graph	If $A \neq C$ **and** they have the **same sign** (i.e. $AC > 0$), the conic is an ellipse. If $\lvert A \rvert > \lvert C \rvert$ then it takes the shape If $\lvert A \rvert < \lvert C \rvert$ then it takes the shape	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ • For a vertical ellipse, $a^2 < b^2$. • For a horizontal ellipse, $a^2 > b^2$. • centre (h, k)
Hyperbola	 Conic is cut at an angle less than the generator angle	 Two intersecting lines	If A **and** C have **different signs** (i.e. $AC < 0$), then the conic is a hyperbola. If A and C are **interchanged**, **and** F remains constant, the direction of opening will change.	• Opening along the x-axis $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ • Opening along the y-axis $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$ • slopes equal to $\pm \frac{b}{a}$ • centre (h, k)
Parabola	 Conic is cut at an angle equal to the generator angle	 Single line (from cylinder - two parallel lines, single line, or no graph)	If A or C, **not both**, equal zero, then the conic is a parabola. If $A \neq 0$ and $C = 0$, then the parabola opens up, (eg. $y = x^2$) or down, (eg. $y = -x^2$) If $A = 0$ and $C \neq 0$, then the parabola opens right, (eg. $x = y^2$) or left, (eg. $x = -y^2$)	• Opening up or down $y - k = a(x-h)^2$ • vertex (h, k) • Opening left or right $x - h = a(y-k)^2$ • vertex (h, k)

Complete Assignment Questions #6 - #11

Assignment

1. Consider the equation $x^2 + y^2 - 6x + 8y + 25 = 0$.

a) Which conic section is suggested by the equation? circle

b) Rewrite the equation in standard form.

$$x^2 - 6x \quad + y^2 + 8y = -25$$
$$x^2 - 6x + 9 + y^2 + 8y + 16 = -25 + 9 + 16$$
$$(x-3)^2 + (y+4)^2 = 0$$

c) Solve the equation in b) to determine the degenerate conic represented by the equation.

point $(3, -4)$

2. Consider the equation $25x^2 + 9y^2 + 50x - 36y + 61 = 0$.

a) Which conic section is suggested by the equation? ellipse

b) Rewrite the equation in standard form.

$$25(x^2 + 2x \quad) + 9(y^2 - 4y \quad) = -61$$
$$25(x^2 + 2x + 1) + 9(y^2 - 4y + 4) = -61 + 25(1) + 9(4)$$
$$25(x+1)^2 + 9(y-2)^2 = 0$$

or

$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{25} = 0$$

c) In what way is the equation in b) different from the usual form for an ellipse.

right side $= 0$

d) An ellipse degenerates to a point. Solve the equation in b) to determine the coordinates of the point represented by the equation.

point $(-1, 2)$

3. Consider the equation $4x^2 - 36y^2 + 16x - 288y - 560 = 0$.

a) Which conic section is suggested by the equation? hyperbola

b) Rewrite the equation in standard form.

$$4(x^2+4x \quad) - 36(y^2+8y \quad) = 560$$
$$4(x^2+4x+4) - 36(y^2+8y+16) = 560 + 4(4) - 36(16)$$
$$4(x+2)^2 - 36(y+4)^2 = 0$$
$$\text{or} \quad \frac{(x+2)^2}{9} - (y+4)^2 = 0$$

c) Solve the equation in b) to determine the degenerate conic represented by the equation.

$$\frac{(x+2)^2}{9} = (y+4)^2$$
$$\frac{x+2}{3} = \pm(y+4)$$
$$x+2 = \pm 3(y+4)$$

$$x+2 = -3(y+4) \qquad x+2 = 3(y+4)$$
$$x+2 = -3y-12 \qquad x+2 = 3y+12$$
$$x+3y+14 = 0 \qquad x-3y-10 = 0$$

Intersecting lines

4. Consider the equation $4x^2 + 49y^2 + 20x - 392y + 809 = 0$.

a) Which conic section is suggested by the equation? ellipse

b) Rewrite the equation in standard form.

$$4(x^2+5x \quad) + 49(y^2-8y \quad) = -809$$
$$4\left(x^2+5x+\frac{25}{4}\right) + 49(y^2-8y+16) = -809 + 4\left(\frac{25}{4}\right) + 49(16)$$
$$4\left(x+\frac{5}{2}\right)^2 + 49(y-4)^2 = 0$$
$$\text{or} \quad \frac{\left(x+\frac{5}{2}\right)^2}{49} + \frac{(y-4)^2}{4} = 0$$

c) Solve the equation in b) to determine the degenerate conic represented by the equation.

point $\left(-\frac{5}{2}, 4\right)$

Multiple Choice

5. Which of the following equations represents a degenerate parabola?

A. $-2x^2 - 2x + 100 = 0$ no term in y
B. $-2x + 2y^2 + 100 = 0$
C. $x^2 - 2y - 100 = 0$
D. $-x^2 - 2y + 100 = 0$

6. If $Ax^2 + Cy^2 - 1 = 0$ represents a circle and $A = 10$, then

A. $C = 0$ $A = C$
B. $C = 10$
C. $C > 10$
D. $C < 10$

7. Which equation represents a vertical ellipse?

A. $-4x^2 - 2y^2 + 100 = 0$ $|A| > |C|$
B. $x^2 + 2y^2 - 100 = 0$
C. $-x^2 + 2y^2 - 100 = 0$
D. $x^2 - 2y^2 + 100 = 0$

8. A hyperbola degenerates into

A. one point
B. one line
C. two parallel lines
D. two intersecting lines

9. If $A = 0$ and $C = 2$ in the equation $Ax^2 + Cy^2 + 8x + 10y - 34 = 0$ then the curve is a parabola opening

A. left
B. right
C. down
D. up

$2y^2 + 8x + 10y - 34 = 0$
$8x = -2y^2 \dots$
of the form $x = -y^2$

10. A quadratic relation is defined by $Ax^2 + Cy^2 + Dx + Ey + F = 0$. If none of the parameters are zero the only shape that is not possible is

A. a circle
B. a hyperbola
C. an ellipse
D. a parabola needs A or $C = 0$

11. The equation $2x^2 + 5y^2 - 10y + 40 = 0$ represents a conic section formed by a plane intersecting a double-napped cone at an angle ellipse

A. equal to the generator angle
B. greater than the generator angle
C. less than the generator angle
D. perpendicular to the axis

Answer Key

1. a) circle **b)** $(x - 3)^2 + (y + 4)^2 = 0$ **c)** the point $(3, -4)$

2. a) ellipse **b)** $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{25} = 0$ **c)** right side of the equation equals 0 and not 1
d) $(-1, 2)$

3. a) hyperbola **b)** $\frac{(x+2)^2}{9} - (y+4)^2 = 0$
c) intersecting lines $x - 3y - 10 = 0$, and $x + 3y + 14 = 0$

4. a) ellipse **b)** $\frac{(x+\frac{5}{2})^2}{49} + \frac{(y-4)^2}{4} = 0$ **c)** the point $(-\frac{5}{2}, 4)$

5. A **6.** B **7.** A **8.** D

9. A **10.** D **11.** B