Permutations and Combinations Lesson #1:
The Fundamental Counting Principle

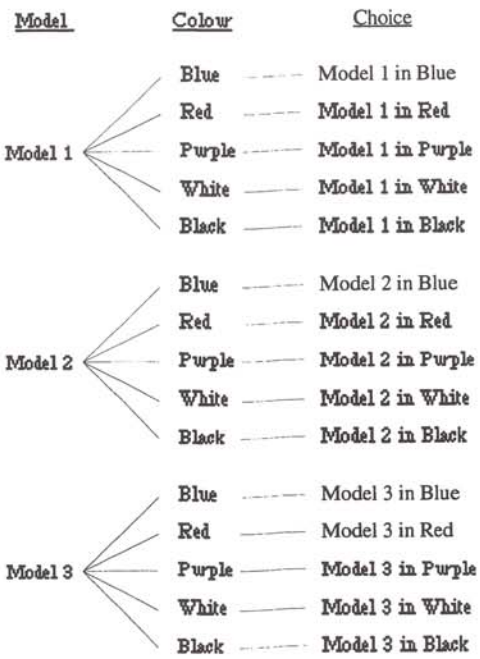
Warm-Up

Blanche is shopping for a car. The dealer tells her that he has 3 different models and that each model is available in 5 different colours.

a) Use the tree diagram to determine how may different choices Blanche has. 15 choices

b) Look at the number of choices there are for the model and the number of choices there are for the colour. How do you use these numbers to arrive at the answer in a)?

(model)(colour)
= (3)(5)
= 15



The Fundamental Counting Principle

Consider a task made up of several stages. If the number of choices for the first stage is a, the number of choices for the second stage is b, the number of choices for the third stage is c, etc., then the number of ways in which the task can be completed is $a \times b \times c \times \ldots$.
This is called the **fundamental counting principle**.

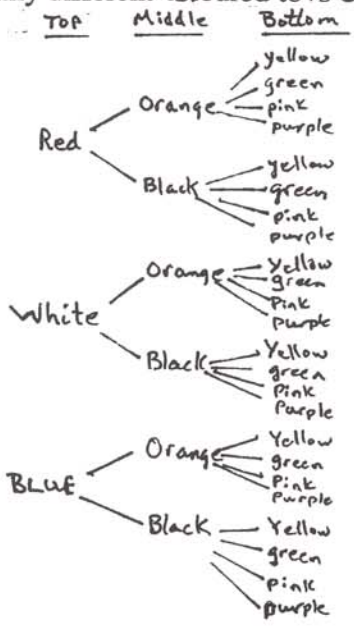
Class Ex. #1

A toy manufacturer makes a wooden toy in three parts:

Part 1: the top part may be coloured red, white or blue,
Part 2: the middle part may be orange or black,
Part 3: the bottom part may be yellow, green, pink or purple.

Determine how many different coloured toys can be produced using:

a) a tree diagram



b) the fundamental counting principle

$\underset{\text{top}}{3} \cdot \underset{\text{middle}}{2} \cdot \underset{\text{bottom}}{4} = 24$ different coloured toys

Determine the number of distinguishable four letter arrangements that can be formed from the word **ENGLISH** if;

a) letters can be repeated? $7 \cdot 7 \cdot 7 \cdot 7 = 7^4$

$= 2401$

b) no letters are repeated and:

i) there are no further restrictions?

$7 \cdot 6 \cdot 5 \cdot 4 = 840$

ii) the first letter must be **E**?

$\underset{E}{1} \cdot 6 \cdot 5 \cdot 4 = 120$

iii) the "word" must contain **G**?

$(\underset{G}{1} \cdot 6 \cdot 5 \cdot 4)4$ OR $\underset{G}{6} \cdot 6 \cdot 5 \cdot 4 = 120$

$6 \cdot \underset{G}{6} \cdot 5 \cdot 4 = 120$

$6 \cdot 5 \cdot \underset{G}{6} \cdot 4 = 120$

$6 \cdot 5 \cdot 4 \cdot \underset{G}{6} = 120$

$(120)(4)$ $= 480$

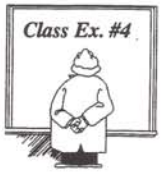
480

iv) the first and last letters must be vowels?

$\underset{V}{2} \cdot 5 \cdot 4 \cdot \underset{V}{1} = 40$

A Math 30 quiz consists of eight multiple choice questions. Each question has four choices A, B, C or D. How many different sets of answers are possible?

$\underset{\substack{A\\B\\C\\D}}{4} \cdot \underset{\substack{A\\B\\C\\D}}{4} \cdot \underset{\substack{A\\B\\C\\D}}{4} \cdot \underset{\substack{A\\B\\C\\D}}{4} \cdot \underset{\substack{A\\B\\C\\D}}{4} \cdot \underset{\substack{A\\B\\C\\D}}{4} \cdot \underset{\substack{A\\B\\C\\D}}{4} \cdot \underset{\substack{A\\B\\C\\D}}{4} = 4^8 = 65\,536$ sets of answers



The telephone numbers allocated to subscribers in a rural area consist of one of the following:

- the digits 345 followed by any three further digits, *or*,
- the digit 2 followed by one of the digits 1 to 5 followed by any three further digits.

How many different telephone numbers are possible?

$\underset{\substack{\text{Area}\\\text{Code}\\345}}{1} \cdot 10 \cdot 10 \cdot 10$ OR $\underset{2}{1} \cdot \underset{1\text{ to }5}{5} \cdot 10 \cdot 10 \cdot 10$

Add

$1000 + 5000$

6000 different telephone numbers

Car number plates in an African country consist of a letter other than I or O followed by three digits, the first of which cannot be zero, followed by any two letters which are not repeated. How many different car number plates can be produced?

$\underset{\substack{\text{letters}\\\neq I\\\neq O}}{24} \cdot \underset{\neq 0}{9} \cdot \underset{\text{digits}}{10} \cdot 10 \cdot \underset{\text{letters}}{26} \cdot 25 = 14\,040\,000$ different car number plates

Consider the digits 2, 3, 5, 6, 7 and 9.

a) If repetitions are not permitted, how many 3-digit numbers can be formed?

$6 \cdot 5 \cdot 4 = 120$

b) How many of these are

i) less than 400?

$\neq 5, 6, 7, 9$

$2 \cdot 5 \cdot 4 = 40$

ii) even?

$5 \cdot 4 \cdot 2 = 40$ (even)

iii) odd?

$5 \cdot 4 \cdot 4 = 80$ (odd)

iv) multiples of 5?

$5 \cdot 4 \cdot 1 = 20$ (5)

Consider a six-digit numeral.(Note: 022713 is classified as the 5-digit numeral 22713).

a) How many odd six-digit numerals have no repeating digits?

$8 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 5$

$\neq 0$ or last digit; last factor: odd

67 200

b) How many even six-digit numerals have no repeating digits? (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

I. (ends with zero)

$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 1 = 15\,120$ (0)

II (even - ends with 2, 4, 6, or 8)

$8 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 53\,760$ (2, 4, 6, 8)

68 880

Complete Assignment Questions #1 - #14

Assignment

1. A football team has the following kit :

jersey: red or black
pants: white, red or black
socks: red or white

If the team plays in a different uniform each week, for how many weeks can it play before it has to repeat a previous uniform?

$2 \cdot 3 \cdot 2 = 12$ (jersey · pants · socks)

2. The score at the end of the second period of a hockey game is: Flames 6 Oilers 3.
How many different possibilities are there for the score at the end of the first period?

$7 \cdot 4 = 28$ (1st: 0-6; 2nd: 0-3)

3. With the new renovations completed at Prestwick High School, there will be seven (7) entrances. In how many different ways can a student coming for Math tutorials;

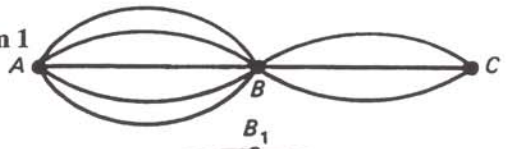
a) enter the school and exit through a different entrance? $7 \cdot 6 = 42$

b) enter and exit through any entrance? $7 \cdot 7 = 49$

c) enter and exit through the same entrance? $7 \cdot 1 = 7$

4. How many ways are there of getting from A to C in each diagram, passing through each point at most once?

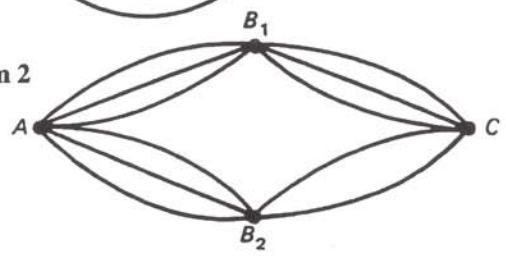
Diagram 1



Answer to Diagram 1

$\frac{5}{AB} \cdot \frac{3}{BC} = 15$

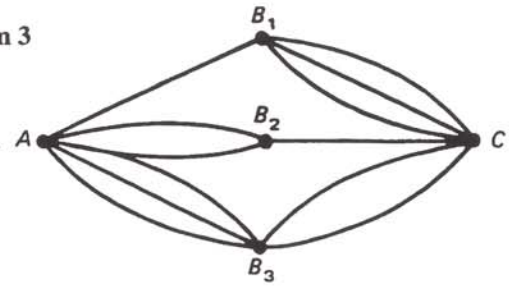
Diagram 2



Answer to Diagram 2

$\frac{3}{AB_1} \cdot \frac{3}{B_1C} + \frac{3}{AB_2} \cdot \frac{2}{B_2C} = 15$

Diagram 3



Answer to Diagram 3

$\frac{1}{AB_1} \cdot \frac{3}{B_1C} + \frac{2}{AB_2} \cdot \frac{1}{B_2C} + \frac{3}{AB_3} \cdot \frac{2}{B_3C}$

$3 + 2 + 6$

11

5. Find the number of four letter "words" that can be formed from the letters of the word **PRODUCE** if;

7

a) each letter can only be used once: $7 \cdot 6 \cdot 5 \cdot 4 = 840$

b) each letter can only be used once and the "word" must:

i) contain only consonants?

$4 \cdot 3 \cdot 2 \cdot 1 = 24$

ii) begin and end with a consonant?

$4 \cdot 5 \cdot 4 \cdot 3 = 240$

(4: consonant; 3: consonant)

iii) begin with a vowel?

$\underset{\text{vowel}}{3} \cdot 6 \cdot 5 \cdot 4 = 360$

(3: O, U, E)

iv) contain the letter **P**?

$4\left(1 \cdot 6 \cdot 5 \cdot 4\right) = 480$

(4: P; 1: P)

v) begin with **D** and end with a vowel?

$\underset{D}{1} \cdot 5 \cdot 4 \cdot \underset{\text{vowel}}{3} = 60$

6. In a class of 30 students, how many ways are there of awarding;

a) a first prize, a second prize and a third prize in Mathematics?

$30 \cdot 29 \cdot 28 = 24360$

b) a Mathematics prize, a Chemistry prize and a Physics prize?
(Assume that each of the students is capable of winning any of the prizes.)

$30 \cdot 30 \cdot 30 = 27000$

7. How many even four digit numerals have no repeated digits?

I II

$9 \cdot 8 \cdot 7 \cdot \underset{0}{1}$ A D D $\quad 8 \cdot 8 \cdot 7 \cdot \underset{2,4,6,8}{4}$

$504 + 1792$

2296

8. A vehicle license plate consists of 3 letters followed by 3 digits. How many different license plates are possible if:

a) there are no restrictions on the letters or digits used?
$\underset{\text{letters}}{26 \cdot 26 \cdot 26} \cdot \underset{\text{digits}}{10 \cdot 10 \cdot 10} = (26^3)(10^3) = 17\,576\,000$

b) no letters may be repeated?
$\underset{\text{letters}}{26 \cdot 25 \cdot 24} \cdot \underset{\text{digits}}{10 \cdot 10 \cdot 10} = (15600)(10^3) = 15\,600\,000$

c) the first digit cannot be zero and no digits can be repeated?
$\underset{\text{letters}}{26 \cdot 26 \cdot 26} \cdot \underset{\neq 0}{9} \cdot \underset{\text{digits}}{9 \cdot 8} = 11\,389\,248$

9. **a)** How many different three-digit numerals can be formed from the digits 1, 5 and 8 if the digits cannot be repeated?
$\underset{1,5,8}{3} \cdot 2 \cdot 1 = 6$

b) How many different three-digit numerals can be formed using the digits 1, 3, 5, 7 and 9 if the digits may be repeated?
$5 \cdot 5 \cdot 5 = 125$

c) How many four-digit numerals can be formed from the digits 0, 2 and 3 if the digits may be repeated? (Note: 0223 is classified as the 3-digit numeral 223).
$\underset{\neq 0}{2} \cdot 3 \cdot 3 \cdot 3 = 54$

d) How many different non-zero numerals are possible using some or all of the numerals 0, 1, 2, and 3 if the digits cannot be repeated?

4 digits: $\underset{\neq 0}{3} \cdot 3 \cdot 2 \cdot 1 = 18$ + 3 digits: $\underset{\neq 0}{3} \cdot 3 \cdot 2 = 18$ + 2 digits: $\underset{\neq 0}{3} \cdot 3 = 9$ + 1 digit: $\underset{\neq 0}{3} = 3$ = **48**

10. How many different sums of money can be made from two pennies, four nickels, two quarters, and five dollar coins?

(a)

Coin	choices	total for each
2 pennies	0, 1, 2	3
4 nickels	0, 1, 2, 3, 4	5
2 quarters	0, 1, 2	3
5 Dollar Coins	0, 1, 2, 3, 4, 5	6

(b) Use Fundamental Counting Principle
$\underset{P}{3} \cdot \underset{N}{5} \cdot \underset{Q}{3} \cdot \underset{\$}{6} = 270$
BUT one selection contains no coins at all.
$\therefore 270 - 1 = 269$

11. Mr. and Mrs. McDonald want a family picture taken with their children, Hamish, Flora and James. In how many different ways can all five line up in a straight line for the picture if;

a) there are no restrictions?
$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

b) the parents must be at either end of the line?
$\underset{P_1\ P_2}{2} \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 12$

c) baby James must be in the middle?
$4 \cdot 3 \cdot \underset{\text{James}}{1} \cdot 2 \cdot 1 = 24$

d) the children alternate with the adults?
$\underset{\text{Child}}{3} \cdot \underset{\text{Adult}}{2} \cdot \underset{\text{Child}}{2} \cdot \underset{\text{Adult}}{1} \cdot \underset{\text{Child}}{1} = 12$

12. Determine the number of distinguishable four letter arrangements that can be formed from the word **PRODUCT** given the following restrictions;

a) no restrictions
$7 \cdot 6 \cdot 5 \cdot 4 = 840$

b) the "word" begins with **PR**?
$\underset{P}{1} \cdot \underset{R}{1} \cdot 5 \cdot 4 = 20$

c) the "word" has two vowels in the middle?
$5 \cdot \underset{V}{2} \cdot \underset{V}{1} \cdot 4 = 40$

d) the "word" has two consonants in the middle?
$5 \cdot \underset{C}{5} \cdot \underset{C}{4} \cdot 4 = 400$

13. Ocean going ships have use coloured flags hung vertically for signalling. By changing the order of the coloured flags, the ships can send out different signals.

If ships carry six different coloured flags, one flag of each colour, how many different signals are possible if;

a) all six flags are used?
$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

b) four flags are used?
$6 \cdot 5 \cdot 4 \cdot 3 = 360$

c) at least two flags are used? (Add them up)

2 flags	+	3 flags	+	4 flags	+	5 flags	+	6 flags	
$6 \cdot 5$	+	$6 \cdot 5 \cdot 4$	+	$6 \cdot 5 \cdot 4 \cdot 3$	+	$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$	+	$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	=
30	+	120	+	360	+	720	+	720	= 1950

Numerical Response

14. Sandra is taking an examination which consists of two parts, A and B, with the following instructions.

- Part A consists of three questions and the student must do two.
- Part B consists of four questions and the student must do two.
- Part A must be completed before starting Part B.
- At the end of the exam the student has to list the order in which she attempted the questions.

$= \left(\underset{Q\ 1\ 2\ 3}{3^{A} \cdot 2}\right)\left(\underset{Q\ 1\ 2\ 3\ 4}{4^{B} \cdot 3}\right)$

$= (6)(12)$

$= 72$

The number of different possible orders is 72 .

(Record your answer in the numerical response box from left to right)

7	2		

Answer Key

1. 12 **2.** 28 **3. a)** 42 **b)** 49 **c)** 7

4. Diagram 1 → 15 **Diagram 2** → 15 **Diagram 3** → 11

5. a) 840 **b)** **i)** 24 **ii)** 240 **iii)** 360 **iv)** 480 **v)** 60

6. a) 24360 **b)** 27000 **7.** 2296

8. a) 17 576 000 **b)** 15 600 000 **c)** 11 389 248

9. a) 6 **b)** 125 **c)** 54 **d)** 48 **10.** 269

11. a) 120 **b)** 12 **c)** 24 **d)** 12 **12. a)** 840 **b)** 20 **c)** 40 **d)** 400

13. a) 720 **b)** 360 **c)** 1950 **14.**

7	2		

Permutations and Combinations Lesson #2:
Factorial Notation and Permutations

Factorial Notation

Consider how many ways there are of arranging 6 different books side by side on a shelf.

In this example we have to calculate the product $6 \times 5 \times 4 \times 3 \times 2 \times 1$

In mathematics this product is denoted by 6! ("6 factorial" or "factorial 6")

In general $n! = n(n-1)(n-2)(n-3) \ldots (3)(2)(1)$, where $n \in W$.

Warm-Up #1

a) Use the factorial key on a calculator to evaluate the following. MATH → PRB ↓ 4 !

$6! =$ 720 $\quad 9! =$ 362 880

b) To simplify a quantity like $\frac{10!}{7!}$ there are a variety of approaches:

i) By using the factorial key on your calculator

$$\frac{10!}{7!} = \frac{3\,628\,800}{5040} = 720 \quad \text{or} \quad \frac{10!}{7!} = 720$$

ii) By Cancellation

$$\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7!}{7!} = \frac{10 \times 9 \times 8 \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 720$$

Find the value of: **a)** $\frac{43!}{40!} = \frac{(43)(42)(41)(40!)}{40!}$ $= 74\,046$ **b)** $\frac{37!}{33!\,4!} = \frac{(37)(36)(35)(34)(33!)}{(33!)(4!)}$ $= 66\,045$

Simplify the following expressions

a) $\frac{n!}{(n-2)!}$

$\frac{n(n-1)(n-2)!}{(n-2)!}$

$n(n-1)$

$n^2 - n$

b) $\frac{(n+3)!}{n!}$

$\frac{(n+3)(n+2)(n+1)n!}{n!}$

$(n+3)(n+2)(n+1)$

c) $\frac{n!}{n(n-1)}$

$\frac{n(n-1)(n-2)!}{n(n-1)}$

$(n-2)!$

Complete Assignment Questions #1 - #5

Warm-Up #2

An internet site's access code consists of three digits. Knowing the three digits is not enough to access the site. The digits have to be entered in the correct order. The ***order*** of the ***arrangement*** of the digits is important.

Joe cannot remember the access code, except that it contains the digits 3, 5 and 7.

- List all the arrangements of these three digits that Joe could use to determine the access code.

3, 5, 7 5, 3, 7 7, 3, 5
3, 7, 5 5, 7, 3 7, 5, 3

There are six possible arrangements to consider and only one of them will access the site. This type of arrangement, where the order is important, is called a **permutation**.

Permutations

An **arrangement** of a set of objects in which the **order of the objects is important** is called a **permutation**.

Permutations of "n" different objects taken all at a time

- List all the arrangements of the letters of the word **CAT**.
- Write the number of permutations in factorial notation.

C – A – T
C – T – A
A – C – T
A – T – C
T – A – C
T – C – A

or $3 \cdot 2 \cdot 1$
$= 3!$
$= 6$

This is an example of the following general rule:

The number of permutations of "*n*" different objects taken all at a time is *n*!

How many permutations are there of the letters of the word:

a) **REGINA** (6)
$6! = 720$

b) **KELOWNA** (7)
$7! = 5040$

Permutations of "n" different objects taken "r" at a time $(r \leq n)$

- Use the fundamental counting principle to determine how many three letter arrangements can be made from the letters of the word **GRAPHITE.** 8

$8 \cdot 7 \cdot 6 = 336$

In the example above we have found the number of permutations of 8 (n) objects taken 3 (r) at a time. This is denoted by ${}_8P_3$.

$${}_8P_3 = 8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times 5!}{5!} = \frac{8!}{(8-3)!}$$

This is an example of the following general rule:

The number of permutations of "n" different objects taken "r" at a time is

$$_nP_r = \frac{n!}{(n-r)!}$$

This formula is on the formula sheet

Class Ex. #4

Use the ${}_nP_r$ key on a calculator to evaluate ${}_8P_3$. Verify using factorials.

MATH → PRB
↓ 2 nPr

8 nPr 3 Enter
336

$_nP_r = \frac{n!}{(n-r)!}$

$= \frac{8!}{(8-3)!}$

$= \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5!} = 336$

Defining 0!

If we replace r by n in the above formula we get the number of permutations of n objects taken n at a time. This we know is $n!$

$$_nP_n = n! = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

For this to be equal to $n!$ the value of 0! must be 1.

0! is defined to have a value of 1.

Class Ex. #5

In a South American country, vehicle license plates consist of any 2 different letters followed by 4 different digits. Find how many different license plates are possible by:

a) the fundamental counting principle

$\underset{\text{letters}}{26} \cdot \underset{\text{letters}}{25} \cdot \underset{\text{digits}}{10} \cdot \underset{\text{digits}}{9} \cdot \underset{\text{digits}}{8} \cdot \underset{\text{digits}}{7}$

$= (650)(5040)$

$= 3\,276\,000$ different license plates

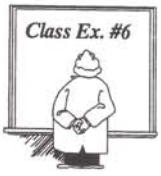
b) permutations

(letters)(digits)

$({}_{26}P_2)({}_{10}P_4)$

$(650)(5040)$

$3\,276\,000$



Solve for n in the equation ${}_nP_4 = 28 \times {}_{n-1}P_2$

$$\frac{n!}{(n-4)!} = \frac{28(n-1)!}{[(n-1)-2]!}$$

$$\frac{n!}{(n-4)!} = \frac{28(n-1)!}{(n-3)!}$$

$$n!(n-3)! = 28(n-1)!(n-4)!$$

$$n(n-1)!(n-3)! = 28(n-1)!(n-4)!$$

$$n(n-3)! = 28(n-4)!$$

$$n(n-3)(n-4)! = 28(n-4)!$$

$$n(n-3) = 28$$

$$n^2 - 3n - 28 = 0$$

$$(n-7)(n+4) = 0$$

$n-7=0$ $\quad n+4=0$

$n=7$ $\quad n=-4$ inadmissible

- In many cases involving simple permutations, the fundamental counting principle can be used in place of the permutation formulas.

Complete Assignment Questions #6 - #13

Assignment

1. Express as single factorials. (No work required.)

a) $6 \times 5 \times 4 \times 3 \times 2 \times 1$ $\quad 6!$

b) $9 \times 8 \times 7 \times 6!$ $\quad 9!$

c) $(n+2)(n+1)n(n-1)\ldots \times 3 \times 2 \times 1$ $\quad (n+2)!$

2. Find the value of the following:

a) $10!$ $\quad 3628\,800$

b) $\frac{8!}{4!}$ $\quad 1680$

c) $\frac{15!}{10!\,5!}$ $\quad 3003$

d) $\left(\frac{25!}{21!}\right)\left(\frac{7!}{11!}\right) = \frac{(25)(24)(23)(22)(21!)(7!)}{(21!)(11)(10)(9)(8)(7!)} = \frac{303\,600}{7\,920} = \frac{115}{3}$

3. Simplify each expression.

a) $\frac{n!}{n}$ $\quad = \frac{n(n-1)!}{n} = (n-1)!$

b) $\frac{(n-3)!}{(n-2)!}$ $\quad = \frac{(n-3)!}{(n-2)(n-3)!} = \frac{1}{n-2}$

c) $\frac{(n+1)!}{(n-1)!}$ $\quad = \frac{(n+1)(n)(n-1)!}{(n-1)!} = n(n+1)$

d) $\frac{(3n)!}{(3n-2)!}$ $\quad = \frac{3n(3n-1)(3n-2)!}{(3n-2)!} = 3n(3n-1)$

4. Express as a quotient of factorials.

a) $9 \times 8 \times 7 \times 6$

$= \frac{9!}{5!}$

b) $20 \times 19 \times 18$

$= \frac{20!}{17!}$

c) $(n+2)(n+1)n$

$= \frac{(n+2)!}{(n-1)!}$

5. Solve the equation.

a) $\frac{(n+1)!}{n!} = 6$

$\frac{(n+1)(n!)}{n!} = 6$

$n+1 = 6$

$n = 5$

b) $(n+1)! = 6(n-1)!$

$(n+1)(n)(n-1)! = 6(n-1)!$

$n(n+1) = 6$

$n^2+n-6 = 0$

$(n+3)(n-2) = 0$

$n \neq -3$ $\quad n = 2$

inadmissible

c) $\frac{(n+2)!}{n!} = 12$

$\frac{(n+2)(n+1)(n!)}{n!} = 12$

$(n+2)(n+1) = 12$

$n^2+3n+2-12 = 0$

$n^2+3n-10 = 0$

$(n+5)(n-2) = 0$

$n \neq -5$ $\quad n = 2$

d) $\frac{(n+1)!}{(n-2)!} = 20(n-1)$

$\frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!} = 20(n-1)$

$n(n+1) = 20$

$n^2+n-20 = 0$

$(n+5)(n-4) = 0$

$n \neq -5$ $\quad n = 4$

6. How many arrangements are there of the letters:

a) DOG

$3!$

$= 6$

b) DUCK

$4!$

$= 24$

c) SANDWICH

$8!$

$= 40320$

d) CANMORE ?

$7!$

$= 5040$

7. How many five-digit numbers can be made from the digits 2, 3, 4, 7 and 9 if no digit can be repeated? $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ or ${}_5P_5$

$= 120$

8. If ${}_nP_r$ is the number of ways that n objects can be arranged r at a time, explain why ${}_7P_0 = 1$.

${}_7P_0 = \frac{7!}{(7-0)!} = \frac{7!}{7!} = 1$

9. Use a permutation formula to determine how many arrangements there are of

a) two letters from the word **GOLDEN** (6)

$6 \cdot 5$ or ${}_6P_2 = 30$

b) three letters from the word **CHAPTERS** (8)

$8 \cdot 7 \cdot 6$ or ${}_8P_3 = 336$

c) four letters from the word **WEALTH** (6)

$6 \cdot 5 \cdot 4 \cdot 3$ or ${}_6P_4 = 360$

or $4 \cdot 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 + 4 \cdot 3 + 4 =$

10. How many numbers can be made from the digits 2, 3, 4, and 5 if no digit can be repeated? (Hint: consider 4 cases - four-digit numbers, three-digit numbers, two-digit numbers, one-digit numbers.) Add 'em up

4 digits	+	3 digits	+	2 digits	+	one digit	
$_4P_4$	+	$_4P_3$	+	$_4P_2$	+	$_4P_1$	
24	+	24	+	12	+	4	= 64

11. Solve each equation, where n is an integer.

a) $\frac{n!}{84} = {}_{n-2}P_{n-4}$

$\frac{n!}{84} = \frac{(n-2)!}{[(n-2)-(n-4)]!}$

$\frac{n!}{84} = \frac{(n-2)!}{(n-2-n+4)!}$

$\frac{n!}{84} = \frac{(n-2)!}{2!}$

$2!\,n! = 84(n-2)!$

$n! = 42(n-2)!$

$n(n-1)(n-2)! = 42(n-2)!$

$n^2 - n - 42 = 0$

$(n+6)(n-7) = 0$

$n \neq -6$ $\quad n = 7$

b) ${}_nP_4 = 8({}_{n-1}P_3)$

$\frac{n!}{(n-4)!} = \frac{8(n-1)!}{(n-1-3)!}$

$\frac{n!}{(n-4)!} = \frac{8(n-1)!}{(n-4)!}$

$n!(n-4)! = 8(n-1)!(n-4)!$

$n! = 8(n-1)!$

$n(n-1)! = 8(n-1)!$

$n = 8$

Multiple Choice **12.** In a ten-team basketball league, each team plays every other team twice, once at home and once away. The number of games that are scheduled is

Home / Away

$10 \cdot 9$ or $_{10}P_2 = 90$

A. 45
B. 90
C. 100
D. 180

Numerical Response **13.** In a competition on the back of a cereal packet, seven desirable qualities for a kitchen (eg. spaciousness, versatility etc.) must be put in order of importance. The number of different entries that must be completed in order to ensure a winning order is $7! = 5040$

(Record your answer in the numerical response box from left to right)

5	0	4	0

Answer Key

1. **a)** 6! **b)** 9! **c)** $(n+2)!$ **2.** **a)** 3 628 800 **b)** 1680 **c)** 3003 **d)** $\frac{115}{3}$

3. **a)** $(n-1)!$ **b)** $\frac{1}{n-2}$ **c)** $n(n+1)$ **d)** $3n(3n-1)$ **4.** **a)** $\frac{9!}{5!}$ **b)** $\frac{20!}{17!}$ **c)** $\frac{(n+2)!}{(n-1)!}$

5. **a)** $n=5$ **b)** $n=2$ **c)** $n=2$ **d)** $n=4$

6. **a)** 6 **b)** 24 **c)** 40 320 **d)** 5040

7. 120 **8.** ${}_7P_0 = \frac{7!}{(7-0)!} = \frac{7!}{7!} = 1$ **9.** **a)** 30 **b)** 336 **c)** 360

10. 64 **11.** **a)** $n=7$ **b)** $n=8$ **12.** B **13.**

5	0	4	0

Permutations and Combinations Lesson #3:
Permutations with Restrictions; Permutations with Repetitions

Permutations with Restrictions

In many problems restrictions are placed on the order in which objects are arranged.
In this type of situation deal with the restrictions first.

In how many ways can all of the letters of the word **ORANGES** be arranged if:

a) there are no further restrictions?

$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 7!$ or ${}_7P_7 = \underline{5040}$

b) the first letter must be an **N**?

$1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ (N)

$= (1)(6!)$ or ${}_1P_1 \cdot {}_6P_6 = \underline{720}$

c) the vowels must be together in the order **O**, **A**, and **E**?

OAE (1 unit) | R (2 units) | N (3 units) | G (4u) | S (5u)

$\therefore$ 5 units $= 5! = \underline{120}$

or

OAE: $1 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= (1)(4!)(5)$

$= \underline{120}$

In how many of the arrangements of the letters of the word **BRAINS** are the vowels together?

$1_A \cdot 1_I \cdot 4 \cdot 3 \cdot 2 \cdot 1$ (5 units)

$+ 1_I \cdot 1_A \cdot 4 \cdot 3 \cdot 2 \cdot 1$ (5 units)

or

$2_{A,I} \cdot 1_I \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$\therefore 5! + 5! = \underline{240}$

$(2!)(4!)(5) = \underline{240}$

Find the number of permutations of the letters in the word **KITCHEN** if:

a) the letters **K**, **C**, and **N** must be together but not necessarily in that order

$3_K \cdot 2_C \cdot 1_N$ (1 unit) | I (2) | T (3) | H (4) | E (5) — 5 units

$\therefore (3!)(5!) = \underline{720}$

or

$K \; C \; N \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$(3!)(4!)(5) = \underline{720}$

b) the vowels must not be together

no restrictions − vowels kept together

$7! - 2_E \cdot 1_E \{ K \; T \; C \; H \; N$ (6 units)

$7! - (2!)(6!)$

$5040 - 1440 = \underline{3600}$

In how many ways can 3 girls and 4 boys be arranged in a row if no two people of the same gender can sit together?

$4_B \cdot 3_G \cdot 3_B \cdot 2_G \cdot 2_B \cdot 1_G \cdot 1_B =$

(Boys)(Girls)

$(4!)(3!)$

$(24)(6) = \underline{144}$

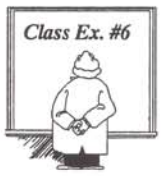
Six actors and eight actresses are available for a play with four male roles and three female roles. How many different cast lists are possible?

$(\underbrace{6 \cdot 5 \cdot 4 \cdot 3}_{\text{MALES}})(\underbrace{8 \cdot 7 \cdot 6}_{\text{FEMALES}}) =$

or

$({}_6P_4)({}_8P_3)$

$(360)(336) = \underline{120960}$



How many odd five-digit numbers can be made with the following digits if no digits can be repeated?

a) 1, 2, 3, 4, 5, 6 and 7

$\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{4} = 1440$
(5; odd: 1, 3, 5, 7)

b) 0, 2, 3, 4, 5, 6 and 7

$\underline{5} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{3} = 900$
($\neq 0$; odd: 3, 5, 7)

Complete Assignment Questions #1 - #7

Permutations With Repetitions

In the previous section, the objects in each set were all different. But what happens when there are letters which repeat within the same word? To examine this scenario, consider the following four letter permutations of a word without repetitive letters, ROSE. Notice there are ${}_4P_4$, or 24 different arrangements.

ROSE	REOS	OSRE	SROE	SERO	EORS
ROES	RESO	OSER	SREO	SEOR	EOSR
RSOE	ORSE	OERS	SORE	EROS	ESRO
RSEO	ORES	OESR	SOER	ERSO	ESOR

Now, if we change the E in ROSE to a S, we get ROSS, a word with two letters which are repeating. If we change all the E's in the above list to S's, we will get all the arrangements for ROSS as shown in the list below.

ROSS ROSS	RSOS RSSO	OSRS OSSR	SROS SRSO	SSRO SSOR	SORS SOSR
RSOS RSSO	ORSS ORSS	OSRS OSSR	SORS SOSR	SROS SRSO	SSRO SSOR

Notice that the 24 original arrangements now become 12 different arrangements - 12 matching pairs of 2 four-letter arrangements. Arrangements like ROSE and ROES from the first list both become ROSS in the second list and count as only one arrangement.

There are $\frac{1}{2}$ or $\frac{1}{2!}$ as many permutations of ROSS as there are of ROSE.

Hence, the number of permutations of ROSS is $\frac{4!}{2!}$, or 12.

If we change the O and E in ROSE to S, we get RSSS, a "word" with three repeating letters, with the arrangements shown below.

RSSS	RSSS	SSRS	SRSS	SSRS	SSRS
RSSS	RSSS	SSSR	SRSS	SSSR	SSSR
RSSS	SRSS	SSRS	SSRS	SRSS	SSRS
RSSS	SRSS	SSSR	SSSR	SRSS	SSSR

Notice the 24 original arrangements now become 4 different arrangements - 4 matching sets of 6 four-letter arrangements. Arrangements like ROSE, ROES, RSOE, RSEO, REOS, and RESO from the first list all become one arrangement of RSSS.

There are $\frac{1}{6}$ or $\frac{1}{3!}$ as many permutations of RSSS as there are of ROSE.

Hence, the number of permutations of RSSS is $\frac{4!}{3!}$, or 4.

There is a pattern in the above lists If a letter appears twice in a word, we divide the total number of arrangements by 2!. If a letter appears three times in a word we divide the total number of arrangements by 3!.

The following formula gives the number of permutations when there are **repetitions.**

> The number of permutations of n objects, where a are the same of one type, b are the same of another type, and c are the same of yet another type, can be represented by the expression below
>
> $$\frac{n!}{a!b!c!}$$
>
> not on formula sheet

Class Ex. #7

Find the number of permutations of the letters of the word:

a) VANCOUVER (9)

2V $\frac{9!}{2!} = \underline{181\,440}$

b) MATHEMATICAL (12)

2M, 3A, 2T $\frac{12!}{(2!)(3!)(2!)} = \underline{19\,958\,400}$

Class Ex. #8

How many arrangements of the word **POPPIES** (7) can be made under each of the following conditions?

a) without restrictions

POPPIES

3P $\frac{7!}{3!} = \underline{840}$

b) if each arrangement begins with a **P**

P|OPPIES

Fix 'P' $\therefore \frac{6!}{2!} = \underline{360}$ OR $\frac{\underset{P}{3}\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3!} = \frac{(3)(6!)}{3!} = \underline{360}$

c) if the first two letters are **P**

$\frac{\underset{P}{3}\cdot\underset{P}{2}\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3!} = \frac{(6)(5!)}{3!} = \underline{120}$ OR $\underset{\text{Fix P}}{\underset{1}{PP}}\cdot\underset{O}{5}\cdot\underset{P}{4}\cdot\underset{I}{3}\cdot\underset{E}{2}\cdot\underset{S}{1} = (1)(5!) = \underline{120}$

d) if all the **P**'s are to be together

PPP OIES (1 2 3 4 5 units) $= 5! = \underline{120}$ or $\frac{\underset{PPP}{3}\cdot\underset{PP}{2}\cdot\underset{P}{1}\cdot 4\cdot 3\cdot 2\cdot 1}{3!} = \frac{(3!)(4!)(5)}{3!} = \underline{120}$

e) if the first letter is **P** and the next one is not **P**

$= \frac{\underset{P}{3}\cdot\underset{\neq P}{4}\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3!} = \frac{(12)(5!)}{3!} = \underline{240}$ or $\frac{\underset{P}{1}\cdot\underset{OIES}{4}\cdot\underset{P}{5}\cdot\underset{P}{4}\cdot 3\cdot 2\cdot 1}{2!} = (1)(4)\left(\frac{5!}{2!}\right) = \underline{240}$

Class Ex. #9

Brett bought a carton containing 10 mini boxes of cereal. There are 3 boxes of Corn Flakes, 2 boxes of Rice Krispies, 1 box of Coco Pops, 1 box of Shreddies, and the remainder are Raisin Bran. Over a ten day period Brett plans to eat the contents of one box of cereal each morning.

How many different orders are possible if on the first morning he has Raisin Bran?

3 CF, 2 RK, 1 CP, 1 S, ~~3~~ 2 RB

$\therefore \frac{(1)(9!)}{(3!)(2!)(2!)} = \frac{(9)(8)(7)(6)(5)(4)(3!)}{(3!)(2!)(2!)} = 15120$ different orders

Complete Assignment Questions #8 - #15

Assignment

1. How many arrangements could be made of the word:

a) **FATHER** if **F** is first?

$\underset{F}{\underline{1}} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$

$(1)(5!) = \underline{\underline{120}}$

or

$(1)({}_5P_5) = \underline{\underline{120}}$

b) **UNCLE** if **C** is first and **L** is last?

$\underset{C}{\underline{1}} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underset{L}{\underline{1}}$

$(1)(3!)(1) = \underline{\underline{6}}$

c) **DAUGHTER** if **UG** is last?

$\underset{D}{\underline{\ }} \mid \underset{A}{\underline{\ }} \mid \underset{H}{\underline{\ }} \mid \underset{T}{\underline{\ }} \mid \underset{E}{\underline{\ }} \mid \underset{R}{\underline{\ }} \mid \underset{UG}{\underline{1}}$

6 units 1 unit

$(6!)(1) = \underline{\underline{720}}$

d) **MOTHER** if the vowels are first and last?

$\underset{O}{\underline{1}} \cdot \underset{Add}{\underline{MTHR}} \cdot \underset{E}{\underline{1}} = 4! = 24$

$+$

$\underset{E}{\underline{1}} \cdot \underline{MTHR} \cdot \underset{O}{\underline{1}} = 4! = 24$

$\underline{\underline{48}}$

2. How many arrangements of the following words can be made if all the vowels must be kept together?

a) **FATHER**

(AE) FTHR $+$ $5!$

(EA) FTHR $5!$

$\underline{\underline{240}}$

or $(2)(5!)$

b) **DAUGHTER**

(AUE) D G H T R

$[(3!)(5!)](6) = \underline{\underline{4320}}$

c) **UNCLE**

(UE) NCL $+$ $4!$

(EU) NCL $4!$

$\underline{\underline{48}}$

or $[(2!)(3!)]4$

d) **EQUATION**

(EUAIO) Q T N =

$= [(5!)(3!)](4)$

$= \underline{\underline{2880}}$

3. Find the number of different arrangements of the letters in the word **ANSWER** under each condition:

a) without restrictions

$= 6!$ or ${}_6P_6$

$= \underline{\underline{720}}$

b) that begin with an **S**

$\underset{S}{\underline{1}} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = (1)(5!) = \underline{\underline{120}}$

c) that begin with a vowel and end with a consonant

$\underset{\text{VOWELS A/E}}{\underline{2}} \cdot \underline{\ \ 4!\ \ } \cdot \underset{\text{Consonants N, S, W, R}}{\underline{4}} = (2)(4!)(4) = \underline{\underline{192}}$

d) that have the three letters **A**, **N**, and **S** adjacent and in the order **ANS**

(ANS) | $\underset{\text{4 units}}{\underline{W}}$ $\underline{E}$ $\underline{R}$

$4! = \underline{\underline{24}}$

OR

$\underset{ANS}{\underline{1}} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = [(1)(3!)]4$

$= \underline{\underline{24}}$

e) that have the three letters **A**, **N**, and **S** adjacent but not necessarily in that order

$\underline{ANS}$ $\underline{W}$ $\underline{E}$ $\underline{R}$

$= (4!)(3!)$

$= \underline{\underline{144}}$

4. How many even four-digit numbers can be made from the digits 0, 2, 3, 4, 5, or 7 if no digit can be repeated?

I — Ends with "0": $5 \cdot 4 \cdot 3 \cdot 1$ (0)

Add +

II — Does not end with "0": $4 \cdot 4 \cdot 3 \cdot 2$ ($\neq 0$)

$60 + 96 = \underline{156}$

5. Ann, Brian, Colin, Diane and Eric go to watch a movie and sit in 5 adjacent seats. In how many ways can this be done under each condition?

a) without restrictions?

$5! = \underline{120}$

b) if Brian sits next to Diane?

(BD) A C E + (DB) A C E

4 units 4 units

$4! + 4! = \underline{48}$

c) if Ann refuses to sit next to Eric?

(i) no restrictions from part (a) = 120

(ii) with Ann next to Eric (b) $-$ 48

(iii) Ann refuses to sit next to Eric 72

6. In how many ways can four adults and five children be arranged in a single line under each condition?

a) without restrictions

9 people

$9! = \underline{362\,880}$

b) if children and adults are alternated

C A

CA CA CA CA C

5! ways arranging children

4! ways arranging adults

$\therefore (5!)(4!) = \underline{2880}$

c) if the adults are all together and the children are all together

$(A_1A_2A_3A_4)(C_1C_2C_3C_4C_5)$

$(4!)(5!) = 2880$

Add

$(C_1C_2C_3C_4C_5)(A_1A_2A_3A_4) = \underline{5760}$

$(5!)(4!) = 2880$

d) if the adults are all together

$(A_1A_2A_3A_4)\,C_1\,C_2\,C_3\,C_4\,C_5$

6 units

$(4!)(6!) = \underline{17280}$

7. Fifteen rugby players line up for a team picture, with seven players in the front row and eight players in the back row. How many different arrangements are possible under each conditions? Express the answer in factorial notation.

a) without restrictions

$\underline{15!}$

b) if the captain is in the middle of the front row

$\underline{14!}$

8. How many different arrangements can be made using all of the letters of each word?

a) **COCHRANE** (8)

$\frac{8!}{2!} = \underline{20\,160}$

b) **WINNIPEG** (8)

$\frac{8!}{2!2!} = \underline{10\,080}$

c) **RED DEER** (7)

$\frac{7!}{3!2!2!} = \underline{210}$

d) **MILLARVILLE** (11)

$\frac{11!}{2!4!} = \underline{831\,600}$

9. Find the number of arrangements of the letters of the word **TATTOO** under each condition:

a) begins with a **T**

T | ATTOO

$= \frac{5!}{2!2!}$

$= 30$

b) begins with two **T**'s

TT | ATOO

$= \frac{4!}{2!}$

$= 12$

c) begins with three **T**'s

TTT | AOO

$= \frac{3!}{2!}$

$= 3$

d) begins with one **T** and the next letter is not a **T**

TA | TTOO Add TO | TTOA

$= \frac{4!}{2!2!} + \frac{4!}{2!}$

$= 6 + 12 = 18$

e) begins with exactly two **T**'s

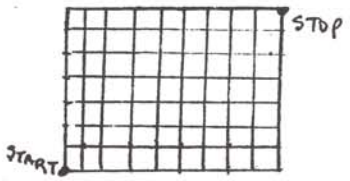
TTO | TAO Add TTA | TOO

$= 3! + \frac{3!}{2!}$

$= 6 + 3$

$= 9$

10. A town has 10 streets running from north to south and 8 avenues running from west to east. A man wishes to drive from the extreme south-west intersection to the extreme north-east intersection, moving either north or east along one of the streets or avenues. Find the number of routes he can take. (This question will be solved in a different manner in a lesson 7 of this unit).



$\frac{16!}{(9!)(7!)} = 11440$

11. How many permutations are there of the letters of the word **MONOTONOUS** under each condition?

a) without restrictions

$\frac{10!}{(4!)(2!)} = 75\,600$

b) if each arrangement begins with a **T**

T | 4 O, 2 N, 1 M, 1 U, 1 S

$= \frac{9!}{4!2!} = 7560$

c) if each arrangement begins with an **O**

O | 3 O, 2 N, 1 M, 1 T, 1 U, 1 S

$= \frac{9!}{3!2!} = 30\,240$

d) if the four **O**'s are to be together

OOOO { N { N { M { T { U { S

$= \frac{7!}{2!} = 2520$

12. A race at the Olympics has 8 runners. In how many orders can their countries finish if there are:

a) 2 Canadian, 1 Russian, 1 German, 1 South African and 3 American athletes.

$\frac{8!}{2!3!} = 3\,360$

b) 1 Canadian, 2 British, 2 Ethiopian, 1 Algerian and 2 Kenyan runners

$\frac{8!}{2!2!2!} = 5040$

13. Naval signals are made by arranging coloured flags in a vertical line and the flags are then read from top to bottom. How many signals using six flags can be made if you have:

a) 3 red, 1 green and 2 blue flags?

$\frac{6!}{3!2!} = 60$

b) 2 red, 2 green and 2 blue flags?

$\frac{6!}{2!2!2!} = 90$

c) unlimited supplies of red, green and blue flags?

$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6 = 729$

RED R R R R R
GREEN G G G G G
BLUE B B B B B

Multiple Choice **14.** The number of different arrangements can be made using all the letters of the word **SASKATOON** is

A. 720

B. 45 360

C. 362 880

D. 725 760

9
2S
2A
2O

$= \frac{9!}{(2!)(2!)(2!)}$

$= 45\,360$

Numerical Response **15.** The number of different ways that seven basketball players can be seated on a bench so that two specified players are always sitting side by side is 1440.

(Record your answer in the numerical response box from left to right)

1	4	4	0

7
AB
BA
— — — — —
1 unit 2 3 4 5 6 units

$(6!)(2) = 1440$

Answer Key

1. **a)** 120 **b)** 6 **c)** 720 **d)** 48

2. **a)** 240 **b)** 4320 **c)** 48 **d)** 2880

3. **a)** 720 **b)** 120 **c)** 192 **d)** 24 **e)** 144

4. 156 **5.** **a)** 120 **b)** 48 **c)** 72

6. **a)** 362 880 **b)** 2880 **c)** 5760 **d)** 17 280

7. **a)** 15! **b)** 14! **8.** **a)** 20 160 **b)** 10 080 **c)** 210 **d)** 831 600

9. **a)** 30 **b)** 12 **c)** 3 **d)** 18 **e)** 9

10. $\frac{16!}{9! \cdot 7!} = 11\,440$ **11.** **a)** 75 600 **b)** 7560 **c)** 30 240 **d)** 2520

12. **a)** 3360 **b)** 5040 **13.** **a)** 60 **b)** 90 **c)** $3^6 = 729$ **14.** B **15.**

1	4	4	0

Permutations and Combinations Lesson #4:
Combinations Part 1

Warm-Up #1

As part of the grade 12 English course, students are required to read the following three books in a three month period:

"The Grapes of Wrath", "The Wars", "The Bean Trees".

a) Due to previous late returns Steve is only allowed to sign out one English book from the school library per month.
List all the different orders in which Steve could sign out the books.

- The Grapes of Wrath, The Wars, The Bean Trees
- The Grapes of Wrath, The Bean Trees, The Wars
- The Wars, The Bean Trees, The Grapes of Wrath
- The Wars, The Grapes of Wrath, The Bean Trees,
- The Bean Trees, The Grapes of Wrath, The Wars
- The Bean Trees, The Wars, The Grapes of Wrath

6

b) Tariq is allowed to sign out all three books at the same time.
How many different ways can he sign out all three books at the same time?

The Grapes of Wrath, The Wars, The Bean Trees

1

Part a) is an example of a permutation where the order is important.
Part b) is an example of a combination where the order is NOT important.

Combinations

A **selection** of a set of objects in which the **order of the selection is NOT important** is called a **combination**.

Warm-Up #2

Suppose that the students in Warm-Up #1 were required to read only two of the books.

a) Complete the table to show the number of ways in which Steve and Tariq could do this:

Steve (Permutations)	Tariq (Combinations)
The Grapes of Wrath, The Wars	The Grapes of Wrath, The Wars
The Grapes of Wrath, The Bean Trees	
The Wars, The Grapes of Wrath	
The Bean Trees, The Grapes of Wrath	The Bean Trees, The Grapes of Wrath
The Wars, The Bean Trees	The Wars, The Bean Trees
The Bean Trees, The Wars	

6 Permutations

3 combinations

b) Complete the following statement:

- The number of combinations is equal to the number of permutations divided by 2 or 2 factorial.

Warm-Up #3

5 A B C D E

Five students, Al, Byron, Colin, Dave and Eric take part in a cross country race to represent their school.

a) Suppose the winner of the race wins \$50, the runner-up wins \$25 and third place is \$10. The table below shows all possible ways in which the three prizes could be awarded to the five participants in the race.

"A" stands for Al, "B" for Byron, "C" for Colin, "D" for Dave, "E" for Eric.

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

- Is this an example of permutations or combinations? Permutations
- How many ways are there to award the three prizes? ${}_5P_3 = 60$

b) For participating in the cross-country race, the school has been awarded three places at a running clinic. The school coach decides to select the 3 lucky students from the ones who took part in the cross country race.

- Use the table from a) (which has been duplicated below) to circle the different ways the three students can be chosen.

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE	= 10
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED	
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE	
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC	
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD	
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC	

- Is this an example of permutations or combinations? Combinations
- How many ways are there to select the three students? $\frac{60}{6}$ or $\frac{{}_5P_3}{3!} = 10$

c) Complete the following statement:

- The number of combinations is equal to the number of permutations divided by 6 or 3 factorial.

Combinations of "n" different objects taken "r" at a time $(r \le n)$

The Warm-Ups are examples of the following general rule:

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!} $$

The number of combinations of "n" items taken "r" at a time is

$$ {}_nC_r = \frac{n!}{(n-r)!\,r!} $$

This formula is on the formula sheet

- The ${}_nC_r$ key on a calculator can be used to evaluate combinations. MATH → PRB → 3 ${}_nC_r$
- In some texts ${}_nC_r$ is written as $\binom{n}{r}$.

Three students from a class of ten are to be chosen to go on a school trip.

a) In how many ways can they be selected?
Write the answer in factorial notation and evaluate.

$${}_{10}C_3 = \frac{10!}{(10-3)!\,3!} = \frac{10!}{7!\,3!} = \frac{10\cdot 9\cdot 8\cdot 7!}{7!\cdot 3\cdot 2\cdot 1} = \frac{720}{6} = 120$$

b) Confirm the answer in a) using the ${}_nC_r$ key on a calculator.

${}_{10}C_3 = 120$

To win the LOTTO 649 a person must correctly choose six numbers from 1 to 49. Jasper, wanting to play the LOTTO 649, began to wonder how many numbers he could make up. How many choices would Jasper have to make to ensure he had the six winning numbers?

${}_{49}C_6 = 13\,983\,816$

The Athletic Council decides to form a sub-committee of seven council members to look at how funds raised should be spent on sports activities in the school. There are a total of 15 athletic council members, 9 males and 6 females. The sub-committee must consist of exactly 3 females.

a) In how many ways can the females be chosen?

Females: ${}_6C_3 = 20$

b) In how many ways can the males be chosen?

Males: ${}_9C_4 = 126$

c) In how many ways can the sub-committee be chosen?

Exactly 3 females
(Females)(Males)
$= ({}_6C_3)({}_9C_4)$
$= (20)(126)$
$= 2520$

d) In how many ways can the sub-committee be chosen if Bruce, the football coach must be included?

Exactly 3 Females
(Females) · (Males) · (Bruce)
$({}_6C_3)\cdot({}_8C_3)\cdot({}_1C_1)$
$= (20)(56)(1)$
$= 1120$

A standard deck of 52 cards has the following characteristics:

- 4 suits (spades, clubs, diamonds, and hearts).
- Each suit has 13 cards.
- Two suits are black (spades and clubs).
- Two suits are red (diamonds and hearts).
- Face cards are considered to be Jacks, Queens, and Kings. 12 cards

Poker is a card game played from a deck of 52 cards. After the cards are dealt, the player may reorder them, therefore order is not important.

a) How many different five card poker hands are possible?

$_{52}C_5 = 2\,598\,960$ different five card poker hands.

b) In how many of the hands in a) will there be:

i) all diamonds? (13)

$_{13}C_5 = 1287$

ii) 4 black cards (26) and 1 red card (26)?

$(_{26}C_4)(_{26}C_1)$

$(14950)(26)$

$388\,700$

iii) 3 kings and 2 queens?

$(_4C_3)(_4C_2)$

$(4)(6)$

24

iv) 3 kings?

and 2 other cards

$(_4C_3)(_{48}C_2)$

$(4)(1128)$

4512

v) four aces?

and 1 other card

$(_4C_4)(_{48}C_1)$

$(1)(48)$

48

vi) 5 cards of the same suit? (called a "flush")

(suits)(5 cards from that suit)

$(4)(_{13}C_5)$

$(4)(1287)$

5148

Complete Assignment Questions #1 - #12

Assignment

1. Pete's Perfect Pizza Company has 9 choices of topping available.

a) How many different 2-topping pizzas can be made? $_9C_2 = 36$

b) How many different 3-topping pizzas can be made? $_9C_3 = 84$

2. A theatre company consisting of ~~6~~ 5 players is to be chosen from ~~15~~ 14 actors. How many selections are possible if the company must include Mrs. Jones?

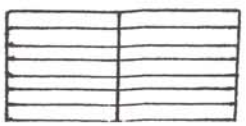
(Mrs. Jones)(Actors)

$(_1C_1)(_{14}C_5)$

$(1)(2002) = 2002$

3. How many different rectangles can be formed by eight horizontal lines and three vertical lines?

(i)



(ii) horizontal · vertical

$(_8C_2)(_3C_2)$

$(28)(3)$

84

4. Edinburgh High School has a twelve-member student council. A four member sub-committee is to be selected to organize dances.

a) How many different sub-committees are possible? $_{12}C_4 = 495$

b) How many four member sub-committees are possible if the council president and vice-president must be members?

(President)(Vice president)(sub committee)
$(_1C_1)(_1C_1)(_{10}C_2)$
$(1)(1)(45) = 45$

5. A basketball coach has five guards and seven forwards on his basketball team.

a) In how many different ways can he select a starting team of two guards and three forwards?

(G)(F)
$(_5C_2)(_7C_3)$
$(10)(35) = 350$

b) How many different starting teams are there if the star player, who plays guard, must be included?

(STAR)(G)(F)
$(_1C_1)(_4C_1)(_7C_3)$
$(1)(4)(35) = 140$

6. Twelve face cards are removed from a deck of fifty-two cards. From the face cards, three card hands are dealt. Determine the number of distinct three card hands that are possible which include:

a) no restrictions
$_{12}C_3 = 220$

b) 3 kings
$_4C_3 = 4$

c) 1 Queen and 2 kings
$(_4C_1)(_4C_2)(_4C_0)$
$(4)(6)(1)$
$= 24$

d) only 1 Jack and 2 other cards
$(_4C_1)(_8C_2)$
$(4)(28)$
$= 112$

7. Consider a standard deck of 52 cards. Determine the number of distinct six card hands that are possible which include:

a) no restrictions
$_{52}C_6$
$= 20\,358\,520$

b) only clubs
$_{13}C_6$
$= 1716$

c) 2 clubs and 4 diamonds
$(_{13}C_2)(_{13}C_4)$
$= (78)(715)$
$= 55\,770$

d) no sevens
$(_4C_0)(_{48}C_6)$
$= 12\,271\,512$

e) 4 tens and 2 others
$(_4C_4)(_{48}C_2)$
$(1)(1128)$
$= 1128$

f) only 1 Jack and 4 Queens and 1 other card
$(_4C_1)(_4C_4)(_{44}C_1)$
$(4)(1)(44)$
$= 176$

8. Explain the meaning of ${}_{10}C_2$. Why does ${}_2C_{10}$ not make sense?

- The notation nCr is used for the number of combinations of 'r' objects taken from 'n' distinct objects.
- Therefore ${}_{10}C_2$ means the number of ways of selecting 2 items from 10. 'ORDER NOT IMPORTANT'
- Cannot select 10 items from 2.

9. Develop a problem where ${}_9C_4$ would be applicable as a solution.

- A mini-mart sells 9 different types of chocolate bars.
- Andrea went into the mini-mart to buy 4 chocolate bars.
- How many selections are possible?

Multiple Choice

10. The number of ways that an executive committee consisting of prime minister, deputy prime minister, treasurer, and secretary can be chosen from 16 student council members is

'Order is important'

A. 4!

B. $\frac{16!}{4!}$

C. $\frac{16!}{12!\,4!}$

D. $\frac{16!}{12!}$

$16 \cdot 15 \cdot 14 \cdot 13$ or ${}_nP_r = \frac{n!}{(n-r)!}$

${}_{16}P_4 = \frac{16!}{(16-4)!}$

$= \frac{16!}{12!}$

11. There are three girls and six boys on the school softball team. The team consists of a pitcher, a catcher, four infielders, and three outfielders. How many ways can the nine different positions be filled if the pitcher must be a girl and the catcher must be a boy?

A. ${}_3C_1 \times {}_6C_1 \times 7!$

B. ${}_3C_1 \times {}_6C_1 \times 9!$

C. ${}_3C_1 \times {}_6C_1$

D. $\frac{9!}{3!\,6!}$

(Pitcher)(Catcher)(4 Infielders)(3 outfielders)
girl Boy

$({}_3C_1)({}_6C_1)(7!)$

Numerical Response

12. Sarah is one of a group of eight people from which a committee of four people must be formed. The number of different committees possible if Sarah must sit on the committee is 35.

(Record your answer in the numerical response box from left to right)

3	5		

(Sarah)$({}_7C_3)$

$({}_1C_1)({}_7C_3)$

$(1)(35) = 35$

Answer Key

1. a) 36 b) 84 **2.** 2002 **3.** 84 **4.** a) 495 b) 45
5. a) 350 b) 140 **6.** a) 220 b) 4 c) 24 d) 112
7. a) 20 358 520 b) 1716 c) 55 770 d) 12 271 512 e) 1128 f) 176
8. The number of ways of selecting 2 items from 10 where the order of selection is not important. You cannot select 10 items from 2.
9. Answers may vary **10.** D **11.** A **12.** 35

Permutations and Combinations Lesson #5:
Combinations Part 2

Combinations Problems with "at least", "at most", etc.

The Student Council decides to form a sub-committee of five council members to look at how funds raised should be spent on the students of the school. There are a total of 11 student council members, 5 males and 6 females.
How many different ways can the sub-committee consist of;

a) exactly three females? 3F · 2M

(Females)(Males)

$({}_6C_3)({}_5C_2)$

$(20)(10)$

200 different ways

b) at least three females? (add 'em up)

3F 2M	+ 4F 1M	+ 5F 0M
$({}_6C_3)({}_5C_2)$	$+({}_6C_4)({}_5C_1)$	$+({}_6C_5)({}_5C_0)$
$(20)(10)$	$+(5)(15)$	$+(6)(1)$
200	+ 75	+ 6

281 different ways

c) at least one female? (add 'em up)

1F 4M	+ 2F 3M	+ 3F 2M	+ 4F 1M	+ 5F 0M
$({}_6C_1)({}_5C_4)$	$+({}_6C_2)({}_5C_3)$	$+({}_6C_3)({}_5C_2)$	$+({}_6C_4)({}_5C_1)$	$+({}_6C_5)({}_5C_0)$
30	+ 150	+ 200	+ 75	+ 6 = 461 different ways

OR

Use COMPLEMENT

Total number of committees possible − number of committees with no FEMALES ie(5 MEN)

$${}_{11}C_5 - {}_5C_5$$

$$462 - 1$$

$$461.$$

Consider a standard deck of 52 cards. How many different five card hands can be formed containing:

a) at least 1 red card?

1R4B + 2R3B + 3R2B + 4R1B + 5R0B

$$({}_{26}C_1)({}_{26}C_4)+({}_{26}C_2)({}_{26}C_3)+({}_{26}C_3)({}_{26}C_2)+({}_{26}C_4)({}_{26}C_1)+({}_{26}C_5)$$

OR

No Restrictions − No RED (all Black)

$${}_{52}C_5 - {}_{26}C_5$$

2 533 180 different five card hands

b) at most 2 kings? 0, 1, 2 Kings

OK 5 other	+ 1K 4 other	+ 2K 3 other
$({}_4C_0)({}_{48}C_5)$	$+({}_4C_1)({}_{48}C_4)$	$+({}_4C_2)({}_{48}C_3)$
$(1)(1\,712\,304)$	$+(4)(194\,580)$	$+(6)(17\,296)$
1 712 304	+ 778 320	+ 103 776

2 594 400

c) exactly two pairs?

1st pair · 2nd pair · 1 other card · what # for the pair

$$({}_4C_2)\cdot({}_4C_2)\cdot({}_{44}C_1)\cdot({}_{13}C_2) = 123\,552$$

Complete Assignment Questions #1 - #6

A class consists of 5 girls and 7 boys. A committee is to be formed consisting of 2 girls and 3 boys. In how many ways can a teacher choose the committee if:

a) there are no further restrictions?

Girls · Boys
$({}_5C_2)({}_7C_3)$
$(10)(35)$
350 ways

b) Johnny, the Principal's son, has to be on the committee?

less 1 boy
Girls · Boys · Johnny
$({}_5C_2)({}_6C_2)({}_1C_1)$
$(10)(15)(1)$
150 ways

c) the twins, Peter and Paul, cannot both be on the committee?

(i) No Restrictions from (a) 350

(ii) Peter and Paul are on the committee

G · B
$({}_5C_2)({}_5C_1)$
$(10)(5)$
50

(iii) Peter and Paul cannot both be on the committee

350 − 50 = 300 ways

Combinations which are Equivalent

Jane calculated ${}_{10}C_2$ to be 45 arrangements. She then calculated ${}_{10}C_8$ to be 45 arrangements.

a) Explain in words why ${}_{10}C_2 = {}_{10}C_8$.

Choosing 2 objects from 10 is the same as having 8 left or choosing 8 objects from 10 is the same as having 2 left.

b) Use factorial notation to show that ${}_{10}C_2 = {}_{10}C_8$.

${}_{10}C_2$	${}_{10}C_8$
$\frac{10!}{2!\,8!}$	$\frac{10!}{8!\,2!}$
$\frac{10\cdot 9\cdot 8!}{2!\,8!}$	$\frac{10\cdot 9\cdot 8!}{8!\,2!}$
45 =	45

c) Give another two examples of equivalent combinations.

(i) ${}_{12}C_3 = {}_{12}C_9$
220 = 220

(ii) ${}_8C_3 = {}_8C_5$
56 = 56

d) Prove the identity ${}_nC_r = {}_nC_{n-r}$.

${}_nC_r$	${}_nC_{n-r}$
$\frac{n!}{(n-r)!\,r!}$	$\frac{n!}{[n-(n-r)!](n-r)!}$
↓	$\frac{n!}{(n-n+r)!\,(n-r)!}$
$\frac{n!}{(n-r)!\,r!}$	$\frac{n!}{(r!)(n-r)!}$

✓ LS = RS ✓

Solving for "n" in Combination Problems

During a Pee Wee hockey tryout, all the players met on the ice after the last practice and shook hands with each other. How many players attended the tryouts if there were 300 handshakes in all?

→ requires 2 people

$$_nC_2 = 300$$

$$\frac{n!}{(n-2)!\,2!} = \frac{300}{1}$$

$$\frac{n!}{(n-2)!} = \frac{600}{1}$$

$$\frac{n(n-1)(n-2)!}{(n-2)!} = \frac{600}{1}$$

$$n(n-1) - 600 = 0$$

$$n^2 - n - 600 = 0$$

$$(n-25)(n+24) = 0$$

$$n = 25 \quad n \neq -24$$

25 players attended the tryouts.

Polygons and Diagonals

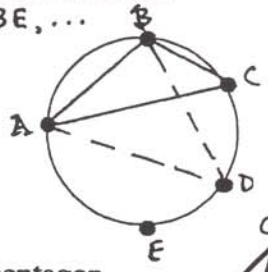
Warm-Up

Consider circles with five points marked on the circumference.

a) How many triangles can be formed using these five points? Write the answer in combination notation.

ABC, ABD, ABE, ...

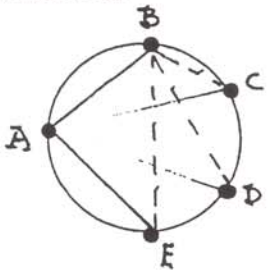
$_5C_3$



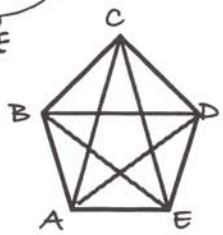
b) How many lines can be drawn connecting two points on the circle? Write the answer in combination notation.

AB, AC, AD, AE
BC, BD, BE
CD, CE
DE

$_5C_2$



c) Sketch a regular pentagon.



i) How many lines can be drawn connecting two points on the pentagon, including the sides of the pentagon? 10

ii) How many of these lines are diagonals of the pentagon? 5

iii) Express the answer to ii) in terms of combinations. $_5C_2 - 5$

d) How many diagonals are there in a regular octagon?

$$_8C_2 - 8$$
$$= 28 - 8$$
$$= 20$$

e) How many diagonals are there in a regular n sided polygon?

$_nC_2 - n$

The number of diagonals in a regular *n*-sided polygon is

$$ {}_nC_2 - n $$

*This formula is **NOT** on the formula sheet*

A polygon has 65 diagonals. How many sides does it have?

$ {}_nC_2 - n = 65 $

$ \frac{n!}{(n-2)!\,2!} = 65 + n $

$ \frac{n(n-1)(n-2)!}{(n-2)!} = \frac{2(65+n)}{1} $

$ n^2 - n = 130 + 2n $

$ n^2 - 3n - 130 = 0 $

$ (n-13)(n+10) = 0 $

$ n = 13 \quad n \neq -10 $

The polygon has 13 sides.

Complete Assignment Questions #7 - #16

Assignment

1. The Athletic Council decides to form a sub-committee of 6 council members to look at a new sports program. There are a total of 15 Athletic Council members, 6 females and 9 males. How many different ways can the sub committee consist of at most one male?

0M 6F + 1M 5F

$ ({}_9C_0)({}_6C_6) + ({}_9C_1)({}_6C_5) $

$ (1)(1) + (9)(6) $

55 different ways

2. A group of 4 journalists is to be chosen to cover a murder trial. There are 5 male and 7 female journalists available. How many possible groups can be formed:

a) consisting of 2 men and 2 women?

MEN · WOMEN

$ ({}_5C_2) \cdot ({}_7C_2) $

$ (10)(21) $

210 possible groups

b) consisting of at least one woman? (add 'em up)

1W 3M + 2W 2M + 3W 1M + 4W 0M

$ ({}_7C_1)({}_5C_3) + ({}_7C_2)({}_5C_2) + ({}_7C_3)({}_5C_1) + ({}_7C_4)({}_5C_0) $

$ (7)(10) + (21)(10) + (35)(5) + (35)(1) = 490 $

OR

No Restrictions − No Women (5 Men)

$ {}_{12}C_4 - {}_5C_4 = 495 - 5 = 490 $

3. Consider a standard deck of 52 cards. How many different four card hands have;

a) at least one black card?

No Restrictions − All Red

$ {}_{52}C_4 - {}_{26}C_4 $

255 775

b) at least 2 kings? (2K 3K 4K)

2K 2 others + 3K 1 0 + 4K 0 other

$ ({}_4C_2)({}_{48}C_2) + ({}_4C_3)({}_{48}C_1) + ({}_4C_4)({}_{48}C_0) $

6768 + 192 + 1

6961

c) two pairs

1st pair · 2nd pair · What number for the pair

$ ({}_4C_2)({}_4C_2)({}_{13}C_2) $

$ (6)(6)(78) $

2808

d) at most 2 clubs? 0, 1, 2 clubs

0 clubs 4 other + 1 Club 3 other + 2 Club 2 other

$ ({}_{13}C_0)({}_{39}C_4) + ({}_{13}C_1)({}_{39}C_3) + ({}_{13}C_2)({}_{39}C_2) $

$ (1)(82\,251) + (13)(9139) + (78)(741) $

82 251 + 118 807 + 57 798

258 856

4. City Council decides to form a sub-committee of five aldermen to investigate transportation concerns. There are 4 males and 7 females. How many different ways can the sub- committee be formed consisting of at least one female member?

No Restrictions − All MEN = At least 1 Female

$_{11}C_5 - {}_4C_5 =$

$462 - 0 =$

462

5. A basketball squad of 11 players is to be chosen from 17 available players. In how many ways can this be done if:

a) Colin and Darryl must be selected?

(C)(D)(other players)

$(_1C_1)(_1C_1)(_{15}C_9)$

$(1)(1)(5005)$

5005

b) Jeff and Brent cannot both be selected?

Neither

No Restrictions − Both on Team

$_{17}C_{11} - 5005$

$12376 - 5005$

7371

6. An all-night showing at a movie theatre is to consist of five movies. There are fourteen different movies available, ten disaster movies and four horror movies. How many possible schedules include:

a) at least one horror movie?

1H4D + 2H3D + 3H2D + 4H1D

$(_4C_1)(_{10}C_4) + (_4C_2)(_{10}C_3) + (_4C_3)(_{10}C_2) + (_4C_4)(_{10}C_1)$

or

No Restrictions − No Disaster = @ least 1 horror

$_{14}C_5 - {}_{10}C_5 =$

$2002 - 252 = 1750$

b) at least four disaster movies?

4D1H + 5D0H

$(_{10}C_4)(_4C_1) + (_{10}C_5)(_4C_0)$

$(210)(4) + (252)(1)$

$840 + 252 = 1092$

c) both "Airport Disaster" and "Halloween Horror"?

(AD)(HH)(others)

$(_1C_1)(_1C_1)(_{12}C_3)$

$(1)(1)(220)$

220

7. The number of ways that a selection of 7 students can be chosen from a class of 28 is the same as the number of ways that n students can be chosen from the same class. What is the value of n?

$_{28}C_7 = {}_{28}C_n \quad \therefore n = 21$

$= {}_{28}C_{21}$

8. How many people are there in a class in which there are 20 ways to select a committee of three people?

$_nC_3 = 20$

$n = 6$

9. Solve for n.

a) $_nC_3 = 84 \quad n = 9$

b) $_{11}C_n = 330$ (two answers)

$n = 4$ or $n = 7$

c) $_nC_7 = {}_{n+1}C_8$

$$\frac{n!}{(n-7)!\,7!} = \frac{(n+1)!}{(n+1-8)!\,8!}$$

$$\frac{n!}{(n-7)!\,7!} = \frac{(n+1)!}{(n-7)!\,8!}$$

$$(n!)(n-7)!\,8! = (n+1)!\,(n-7)!\,(7!)$$

$$(n!)(8)(7!) = (n+1)(n!)(7!)$$

$$8 = n+1$$

$$7 = n$$

10. How many diagonals are there in each?

a) a hexagon 6

$_nC_2 - n$
$= {}_6C_2 - 6$
$= 15 - 6$
$= \underline{9}$

b) a decagon 10

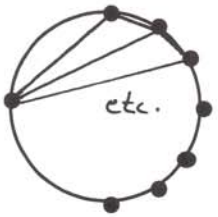
$_nC_2 - n$
$= {}_{10}C_2 - 10$
$= 45 - 10$
$= \underline{35}$

c) an p-sided polygon

$_nC_2 - n$
$= {}_pC_2 - p$
$= \frac{p!}{(p-2)!2!} - p$
$= \frac{p(p-1)(p-2)!}{(p-2)!\,2} - p$
$= \frac{p(p-1)}{2} - p$
$= \frac{p^2-p}{2} - \frac{2p}{2}$
$= \frac{p^2-3p}{2}$
$= \underline{\frac{p(p-3)}{2}}$

11. There are eight visible points on the circle below. How many triangles can be made using these eight points?

$_8C_3 = \underline{56 \text{ triangles}}$



12. If 35 quadrilaterals can be placed on a circle with a series of points on it, then how many points are on the circle?

$_nC_4 = 35$
$n = \underline{7}$

Multiple Choice

13. After everyone had shaken hands once with everyone else in a room, there was a total of 66 handshakes. How many people were in the room?

A. 11
B. 12
C. 33
D. 67

$_nC_2 = 66$
$\underline{n = 12}$

Numerical Response

14. A basketball team consists of some guards and six forwards. If there are 420 ways to select two guards and three forwards to the starting line-up, then the number of guards on the team is 7.

? G 6 F

$(_nC_2)(_6C_3) = 420$
$(_nC_2)(20) = 420$
$_nC_2 = 21$
$\underline{n = 7}$

(Record your answer in the numerical response box from left to right)

7			

15. Collinear points are points which share the same straight line. The number of triangles which can be formed from 10 points if no three of the points are collinear is 120.

(Record your answer in the numerical response box from left to right)

1	2	0	

$_{10}C_3 = \underline{120}$

16. There are 170 diagonals in a polygon. The number of sides of the polygon is 20.

(Record your answer in the numerical response box from left to right)

2	0		

$_nC_2 - n = 170$
$_nC_2 = 170 + n$
$\frac{n!}{(n-2)!2!} = 170 + n$
$\frac{n(n-1)(n-2)!}{(n-2)!} = 340 + 2n$
$n^2 - n - 2n - 340 = 0$
$n^2 - 3n - 340 = 0$
$n^2 - 3n - 340 = 0$
$(n-20)(n+17) = 0$
$\underline{n = 20}$ $\cancel{n = -17}$

Answer Key

1. 55 **2.** **a)** 210 **b)** 490
3. **a)** 255 775 **b)** 6961 **c)** 2808 **d)** 258 856
4. 462 **5.** **a)** 5005 **b)** 7371
6. **a)** 1750 **b)** 1092 **c)** 220 **7.** 21 **8.** 6
9. **a)** 9 **b)** 4 or 7 **c)** 7
10. **a)** 9 **b)** 35 **c)** $\frac{p(p-3)}{2}$ **11.** 56 **12.** 7 **13.** B

14.

7			

15.

1	2	0	

16.

2	0		

Permutations and Combinations Lesson 6:
Pascal's Triangle and the Binomial Theorem

Warm-Up #1

Study the following expansion of $(x+y)^n$, where $0 \le n \le 6$.

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$
$$(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

You may want to do the expansions for yourself to verify the above results.

Properties of the Expansion $(x+y)^n$

Use the above expansions to complete the following:

1. There are $n+1$ terms.
2. The sum of the exponents of x and y in each term is n.
3. The exponents of x decrease term by term from n to 0. (by 1)
4. The exponents of y increase term by term from 0 to n. (by 1)
5. The coefficients in each expansion form a symmetrical arrangement.

Warm-Up #2

The coefficients in the above expansion can be put in a triangular array known as Pascal's Triangle (named after Blaise Pascal who developed applications of the triangle in the seventeenth century.)

a) Use Warm-Up #1 to complete the 7th row of Pascal's Triangle.

b) Use the pattern in the triangle to complete the 8th and 9th row.

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

c) Complete the following expansions:

$(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$

$(x+y)^8 = 1x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + 1y^8$

Warm-Up #3

Compare the combination array below with Pascal's Triangle.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & {}_0C_0 & & & & & & & & \\
 & & & & & {}_1C_0 & & {}_1C_1 & & & & & & & \\
 & & & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & & & & & \\
 & & & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & & & & & \\
 & & {}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4 & & & & \\
 & {}_5C_0 & & {}_5C_1 & & {}_5C_2 & & {}_5C_3 & & {}_5C_4 & & {}_5C_5 & & & \\
{}_6C_0 & & {}_6C_1 & & {}_6C_2 & & {}_6C_3 & & {}_6C_4 & & {}_6C_5 & & {}_6C_6 & & \\
{}_7C_0 & & {}_7C_1 & & {}_7C_2 & & {}_7C_3 & & {}_7C_4 & & {}_7C_5 & & {}_7C_6 & & {}_7C_7 \\
{}_8C_0 & & {}_8C_1 & & {}_8C_2 & & {}_8C_3 & & {}_8C_4 & & {}_8C_5 & & {}_8C_6 & & {}_8C_7 \quad {}_8C_8
\end{array}
$$

$$
\begin{array}{ccccccccccccccc}
 & & & & & & 1 & & & & & & & & \\
 & & & & & 1 & & 1 & & & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 \quad 1
\end{array}
$$

a) Complete the next row in both tables.

b) Pascal derived the following theorem known as Pascal's Theorem.

Verify the theorem for $n = 5$ and $r = 1$.

$$
{}_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}
$$

${}_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}$

${}_{5+1}C_{1+1} = {}_5C_1 + {}_5C_{1+1}$

${}_6C_2 = {}_5C_1 + {}_5C_2$.

$15 \quad | \quad 5 + 10$

$15 = 15$ ✓ ✓

Warm-Up #4 *The Sum of the Numbers in the kth Row of Pascal's Triangle.*

$$
\begin{array}{ccccccccccccccc}
 & & & & & & {}_0C_0 & & & & & & & & \\
 & & & & & {}_1C_0 & & {}_1C_1 & & & & & & & \\
 & & & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & & & & & \\
 & & & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & & & & & \\
 & & {}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4 & & & & \\
 & {}_5C_0 & & {}_5C_1 & & {}_5C_2 & & {}_5C_3 & & {}_5C_4 & & {}_5C_5 & & & \\
{}_6C_0 & & {}_6C_1 & & {}_6C_2 & & {}_6C_3 & & {}_6C_4 & & {}_6C_5 & & {}_6C_6
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \text{Row} \\
 & & & & & 1 & & 1 & & & & & 1 \\
 & & & & 1 & & 2 & & 1 & & & & 2 \\
 & & & 1 & & 3 & & 3 & & 1 & & & 3 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & 4 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & 5 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \quad 6 \quad 7
\end{array}
$$

Complete the table which gives the sum of the numbers in each of the first six rows of Pascal's Triangle.

Row	1	2	3	4	5	6	...	k
Sum	1	2	4	8	16	32		2^{k-1}

a) Complete the following statement: 2^2 2^3 2^4 2^5

"The sum of the numbers in the kth row of Pascal's Triangle is 2^{k-1}."

b) Look at row 7 in the diagram at the top of the page. This row has a sum of 2^6. $= 64$

Using the above rule we can see that

$${}_6C_0 + {}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6 = 2^6 = 64$$
$$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$$

In general,

$$
{}_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \ldots + {}_nC_{n-1} + {}_nC_n = 2^n
$$

Warm-Up #5 — *The Sum of the Coefficients in the Expansion of* $(x+y)^n$

Consider the expansion of $(x+y)^n$.

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

Complete the table which gives the sum of the coefficients in each of the first five expansions of $(x+y)^n$.

n	1	2	3	4	5	...	n
sum of coefficients	2	4	8	16	32		2^n

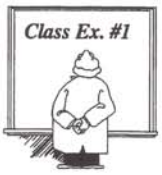
Complete the following statement : 2^1 2^2 2^3 2^4 2^5

"The sum of the coefficients in the expansion of $(x+y)^n$ is 2^n."

Memorize these results.

- The sum of the numbers in the kth row of Pascal's Triangle is 2^{k-1}
- ${}_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \ldots + {}_nC_{n-1} + {}_nC_n = 2^n$
- The sum of the coefficients in the expansion of $(x+y)^n$ is 2^n

---



a) What is the sum of the numbers in the tenth row of Pascal's Triangle? $2^{K-1} = 2^{10-1} = 2^9 = 512$

b) What is the sum of the coefficients in the expansion of $(m+n)^{12}$?
$(x+y)^{12} = 2^n = 2^{12} = 4096$

c) What is the sum of: **i)** ${}_{15}C_0 + {}_{15}C_1 + {}_{15}C_2 + \ldots + {}_{15}C_{15}$?
${}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_n = 2^n = 2^{15} = 32\,768$

ii) ${}_6C_0 + {}_6C_1 + {}_6C_2 + \ldots + {}_6C_5$?
${}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_5 - {}_nC_6$
$= 2^6 - {}_6C_6$
$= 64 - 1$
$= 63$

Complete Assignment Questions #1 - #6

Warm-Up #6 — *Discovering the Binomial Theorem*

Referring to the combination array and Pascal's Triangle in Warm-Up #3, it is not just a coincidence that the combination symbols represent the numbers in Pascal's Triangle.

To investigate this consider the expansion of $(x+y)^6$.

$$(x+y)^6 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$$

$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Notice that each term in the expansion is a *combination* of x's and y's.

Consider how the third term (i.e.$15x^4y^2$) is formed. It is formed by choosing a y from any two of the six factors in the expansion of $(x+y)^6$ and an x from the remaining four factors. The two y's can be chosen in ${}_6C_2$ (or 15) ways.

The four x's can then be chosen in ${}_4C_4$ (or 1) way. This leads to the term ${}_6C_2x^4y^2$.

- Write the coefficients <u>in terms of combinations</u> for the expansion below.

$$(x+y)^6 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$$

$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

$$= {}_6C_0x^6 + {}_6C_1x^5y + {}_6C_2x^4y^2 + {}_6C_3x^3y^3 + {}_6C_4x^2y^4 + {}_6C_5xy^5 + {}_6C_6y^6$$

This leads to the following general formula for binomial expansions known as the Binomial Theorem.

Binomial Theorem

$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_kx^{n-k}y^k + \ldots + {}_nC_ny^n$, where $n \in I, n \geq 0$

General Term of the Expansion of $(x+y)^n$

The term ${}_nC_kx^{n-k}y^k$ is called the **general term** of the expansion.

It is the $(k+1)^{th}$ term in the expansion (not term k)

$$t_{k+1} = {}_nC_k\,x^{n-k}\,y^k$$

on the formula sheet

Find: **a)** the fifth term of $(x+y)^8$ t_5 **b)** the middle term of $(2x-5)^6$. t_4 $[2x+(-5)]^6$

a) $t_{K+1} = {}_nC_K\, x^{n-K} y^K$ 9 terms $n=8$ $K=4$

$t_{4+1} = ({}_8C_4)(x^4)(y^4)$

$t_5 = \underline{70x^4y^4}$

b) $t_{K+1} = {}_nC_K\, x^{n-K} y^K$ 7 terms $n=6$ $K=3$

$t_{3+1} = {}_6C_3(2x)^3(-5)^3$

$= 20(8x^3)(-125)$

$t_4 = \underline{-20000x^3}$

Expand $(3x-2)^3$. $= [\overset{x}{3x} + \overset{y}{(-2)}]^3$ 4 terms $n=3$

$= {}_nC_0x^3 + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + {}_nC_3y^3$

$= {}_3C_0(3x)^3 + {}_3C_1(3x)^2(-2) + {}_3C_2(3x)^1(-2)^2 + {}_3C_3(-2)^3$

$= (1)(27x^3) + (3)(9x^2)(-2) + (3)(3x)(4) + (1)(-8)$

$= \underline{27x^3 - 54x^2 + 36x - 8}$

One term in the expansion of $(x+a)^{10}$ is $3\,281\,250\,x^4$. Determine the numerical value of a, $a>0$. $n=10$ $K=6$

${}_nC_K\, x^{n-K} y^K = 3\,281\,250\,x^4$

${}_{10}C_6\, x^{10-6} y^6 = 3\,281\,250\,x^4$

$210x^4y^6 = 3\,281\,250\,x^4$

$y^6 = 15\,625$

$a = y = \underline{5}$

Find the constant term (i.e. the term independent of x) in the expansion of $\left(2x - \frac{1}{x^2}\right)^{15}$. $n=15$ $K=K$ 16 terms

$\left[\overset{x}{2x} + \overset{y}{\left(\frac{-1}{x^2}\right)}\right]^{15}$

(i) $t_{K+1} = {}_nC_K\, x^{n-K} y^K$

$= {}_{15}C_K(2x)^{15-K}\left(-\frac{1}{x^2}\right)^K$

$= ({}_{15}C_K)(2^{15-K})(x^{15-K})(-1)^K\left(\frac{1}{x^2}\right)^K$

$= ({}_{15}C_K)(-1)^K(x^{15-K})(2^{15-K})(x^{-2})^K$

$= ({}_{15}C_K)(-1)^K(x^{15-K})(2^{15-K})(x^{-2K})$

$= ({}_{15}C_K)(-1)^K(x^{15-3K})(2^{15-K})$

$= ({}_{15}C_K)(-1)^K(2^{15-K})\boxed{(x^{15-3K})}$ ⟹ by definition Exponent = 0

$15-3K = 0$

$15 = 3K$

$5 = K$

Therefore $t_{K+1} = t_{5+1}$

$\underline{t_6}$ is the term independent of 'x'

(ii) Constant term is t_6

$t_6 = {}_nC_K\, x^{n-K} y^K$

$= ({}_{15}C_5)(2x)^{10}\left(-\frac{1}{x^2}\right)^5$

$= (3003)(1024x^{10})\left(-\frac{1}{x^{10}}\right)$

$= \underline{-3\,075\,072}$

Complete Assignment Questions #7 - #13

Assignment

1. Consider the expansion of $(x+y)^n$.

a) What are the coefficients of the first and last terms? 1

b) How many terms are there in the expansion? $n+1$

c) What is the sum of the exponents in each term? n

d) What is the coefficient of the third term?

$n = n$, $K = 2$

$${}_nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

e) The term in which position contains y^4?

$$\underbrace{x^n}_{t_1} + \underbrace{x^{n-1}y}_{t_2} + \underbrace{x^{n-2}y^2}_{t_3} + \underbrace{x^{n-3}y^3}_{t_4} + \underbrace{x^{n-4}y^4}_{t_5}$$ 5th term

2. What are the first three terms in the ninth row of Pascal's Triangle?

${}_8C_0 + {}_8C_1 + {}_8C_2 + \dots$

$1 + 8 + 28 + \dots$

3. What are the last three terms in the sixteenth row of Pascal's Triangle?

$\dots + {}_{15}C_{13} + {}_{15}C_{14} + {}_{15}C_{15}$

$\dots + 105 + 15 + 1$

4. Find the value of n if the expansion of:

a) $(2x+3)^n$ has 18 terms

$n = 17$

b) $(3x-5)^{4n-3}$ has 26 terms.

exponent is 25

$4n - 3 = 25$

$4n = 28$

$n = 7$

5. a) What is the sum of the terms in the ninth row of Pascal's Triangle?

Exponent is 8 $\quad 2^8 = 256$

b) What is the sum of the coefficients in the expansion of $(a+b)^9$? $2^9 = 512$

c) What is the sum ${}_9C_0 + {}_9C_1 + {}_9C_2 + \dots + {}_9C_9$?

${}_nC_0 + {}_nC_1 + {}_nC_2 + \dots + {}_nC_n = 2^n$

$= 2^9 = 512$

6. Find the following sums :

a) ${}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + \dots + {}_{10}C_{10}$ + ${}_{10}C_0$

$2^{10} - {}_{10}C_0$

$= 1024 - 1$

$= 1023$

b) ${}_7C_6 + {}_7C_5 + {}_7C_4 + \dots + {}_7C_1 + {}_7C_0$ + ${}_7C_7$

$= 2^7 - ({}_7C_7 + {}_7C_0)$

$= 128 - (1+1)$

$= 128 - 2$

$= 126$

7. Find the indicated term of each expansion.

a) the fifth term of $(a-b)^5$ $\quad t_5 \quad [a+(-b)^5]$

6 terms

$t_{K+1} = {}_nC_K\, x^{n-K} y^K$ $\quad n = 5$

$t_{4+1} = ({}_5C_4)(a^{5-4})(-b)^4$ $\quad K = 4$

$t_5 = 5ab^4$

b) the second term of $(x-2)^6 = [x+(-2)]^6$ $\quad t_2$

7 terms

$t_{K+1} = {}_nC_K\, x^{n-K} y^K$ $\quad n = 6$

$t_{1+1} = ({}_6C_1)(x^{6-1})(-2)^1$ $\quad K = 1$

$t_2 = 6x^5(-2)$

$t_2 = -12x^5$

t_3

c) the third term of $(3x+2y)^9$ — 10 terms

$t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$ $\quad n=9,\ K=2$

$t_{2+1} = ({}_9C_2)(3x)^7(2y)^2$

$= (36)(2187x^7)(4y^2)$

$t_3 = \underline{314\,928x^7y^2}$

t_4 $\quad [a^2+(-2a)]^7$

d) the fourth term of $(a^2-2a)^7$ — 8 terms

$t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$ $\quad n=7,\ K=3$

$t_{3+1} = ({}_7C_3)(a^2)^4(-2a)^3$

$= (35)(a^8)(-8a^3)$

$t_4 = \underline{-280a^{11}}$

e) the middle term of $\left(2-\frac{x}{2}\right)^6 = \left[2+\left(\frac{-x}{2}\right)\right]^6$ — 7 terms

$t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$ $\quad n=6,\ K=3$

$t_{3+1} = ({}_6C_3)(2)^3\left(\frac{-x}{2}\right)^3$

$= (20)(8)\left(-\frac{x^3}{8}\right)$

$t_4 = \underline{-20x^3}$

8. Expand and write in the simplest form:

a) $(2x+y)^3$ — 4 terms

$= {}_3C_0(2x)^3 + {}_3C_1(2x)^2y + {}_3C_2(2x)(y^2) + {}_3C_3y^3$

$= (1)(8x^3) + (3)(4x^2)y + (3)(2x)y^2 + 1(y^3)$

$= \underline{8x^3 + 12x^2y + 6xy^2 + y^3}$

$\left[x+\left(-\frac{1}{x}\right)\right]^4$

b) $\left(x-\frac{1}{x}\right)^4$ — 5 terms

$= {}_4C_0(x)^4 + {}_4C_1(x)^3\left(-\frac{1}{x}\right) + {}_4C_2(x)^2\left(-\frac{1}{x}\right)^2 + {}_4C_3(x)\left(-\frac{1}{x}\right)^3 + {}_4C_4\left(-\frac{1}{x}\right)^4$

$= (1)(x^4) + (4)(x^3)\left(-\frac{1}{x}\right) + (6)(x^2)\left(\frac{1}{x^2}\right) + (4)(x)\left(-\frac{1}{x^3}\right) + (1)\left(\frac{1}{x^4}\right)$

$= \underline{x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}}$

$[1+(-2x)]^{12}$

9. Find the indicated term of each expansion.

$[3x+(-y)]^7$

a) the term in x^3 in $(1-2x)^{12}$ — 13 terms

$t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$ $\quad n=12,\ K=3$

$t_{3+1} = ({}_{12}C_3)(1)^9(-2x)^3$

$= (220)(1)(-8x^3)$

$t_4 = \underline{-1760x^3}$

b) the term in x^4y^3 in $(3x-y)^7$ — 8 terms

$t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$ $\quad n=7,\ K=3$

$t_{3+1} = ({}_7C_3)(3x)^4(-y)^3$

$= (35)(81x^4)(-y^3)$

$t_4 = \underline{-2835x^4y^3}$

10. a) One term in the expansion of $(x+a)^8$ is $448x^6$. Determine the value of a, $a>0$.

$n=8$, $K=n-6=8-6=2$

${}_nC_K\, x^{n-K}\, y^K = 448x^6$

$({}_8C_2)(x^{8-2})(a)^2 = 448x^6$

$28x^6a^2 = 448x^6$

$28a^2 = 448$

$a^2 = 16$

$\underline{a=4}$

b) One term in the expansion of $(x+b)^{11}$ is $-4455x^8$. Determine the value of b.

$n=11$, $K=n-8=11-8=3$

${}_nC_K\, x^{n-K}\, y^K = -4455x^8$

$({}_{11}C_3)(x^{11-3})(b)^3 = -4455x^8$

$165x^8b^3 = -4455x^8$

$165b^3 = -4455$

$b^3 = -27$

$\underline{b=-3}$

11. Find the indicated term of each expansion.

a) the constant term in the expansion of $\left(x^2 - \frac{1}{x}\right)^6$

7 terms
$n = 6$
$k = K$

(i) $t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$

$t_{K+1} = ({}_6C_K)(x^2)^{6-K}\left(-\frac{1}{x}\right)^K$

$= ({}_6C_K)(x^{12-2K})(-x^{-1})^K$

$= ({}_6C_K)(x^{12-2K})(-1)^K(x^{-K})$

$= ({}_6C_K)(-1)^K(x^{12-2K})(x^{-K})$

'Exponent is 0' $x^{12-3K} = 0$

(ii) $12 - 3K = 0$

$12 = 3K$

$4 = K$

Indicated term $t_{K+1} = t_{4+1}$

$= t_5$

5th term

'Constant term is'

(iii) Value of t_5

$t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$

$t_{4+1} = ({}_6C_4)(x^2)^2\left(-\frac{1}{x}\right)^4$

$t_5 = (15)(x^4)\left(\frac{1}{x^4}\right)$

$t_5 = 15$

b) the term independent of x in $\left(2x + \frac{1}{x^3}\right)^8$

9 terms
$n = 8$
$K = K$

(i) $t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$

$= ({}_8C_K)(2x)^{8-K}(x^{-3})^K$

$= ({}_8C_K)(2^{8-K})(x^{8-K})(x^{-3K})$

x^{8-4K} has Exponent $= 0$

$8 - 4K = 0$

$8 = 4K$

$2 = K$

(ii) Indicated term $= t_{K+1}$

$= t_{2+1}$

$= t_3$

(iii) Constant term is Value t_3

$t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$

$t_{2+1} = ({}_8C_2)(2x)^6\left(\frac{1}{x^3}\right)^2$

$= (28)(64x^6)\left(\frac{1}{x^6}\right)$

$= (28)(64)$

$t_3 = 1792$

Numerical Response

12. If $(2x + 3)^{2n+3}$ has 18 terms, then the value of n is $\underline{7}$.

(Record your answer in the numerical response box from left to right)

Exponent is 17

$2n + 3 = 17$

$2n = 14$

$n = 7$

7			

13. The term in x^{11} in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$ has a numerical coefficient, to the nearest whole number, of $\underline{120}$.

11 terms
$n = 10$ $K = 3$

(Record your answer in the numerical response box from left to right)

1	2	0	

$t_{K+1} = {}_nC_K\, x^{n-K}\, y^K$

$= ({}_{10}C_3)(x^2)^7\left(\frac{1}{x}\right)^3$

$= (120)(x^{14})\left(\frac{1}{x^3}\right)$

$t_4 = 120x^{11}$

Answer Key

1. **a)** 1 **b)** $n + 1$ **c)** n **d)** ${}_nC_2 = \frac{n(n-1)}{2}$ **e)** term 5

2. 1, 8, 28 **3.** 105, 15, 1 **4.** **a)** 17 **b)** 7

5. **a)** 256 **b)** 512 **c)** 512 **6.** **a)** 1023 **b)** 126

7. **a)** $5ab^4$ **b)** $-12x^5$ **c)** $314\,928\,x^7y^2$ **d)** $-280a^{11}$ **e)** $-20x^3$

8. **a)** $8x^3 + 12x^2y + 6xy^2 + y^3$ **b)** $x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}$ **9.** **a)** $-1760x^3$ **b)** $-2835x^4y^3$

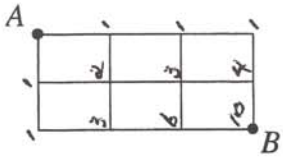
10. **a)** 4 **b)** –3 **11.** **a)** 15 **b)** 1792 **12.** 7 **13.** 120

Permutations and Combinations Lesson #7:
Pathway Problems

Warm-Up

Consider the following problem:

"Find the number of pathways from *A* to *B* if paths must always move closer to *B*"



This problem can be solved in several ways.

a) Solve this problem by tracing the number of paths on the grid.

b) Explain how this problem can be regarded as an example of permutations with repetitions. Determine the number of pathways using this approach.

- To get from A to B, you must travel 5 blocks lengths.
- Any path from A to B is comprised of 3 blocks right (3R) and 2 blocks down (2D)
- same as the number of permutations of a 5-letter word made of 3R and 2D

∴ RRRDD

$= \frac{5!}{3!\,2!}$

= 10 pathways

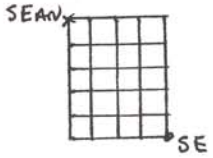
c) Explain how this problem can be regarded as an example of combinations. Use the combination formula or Pascal's Triangle to determine the number of pathways

see sketch

A city centre has a rectangular road system with 5 streets running north to south and 6 avenues running west to east.

N ↑

a) Draw a grid to represent this situation.



b) Sean is driving a car and is situated at the extreme northwest corner of the city centre. In how many ways can he drive to the extreme southeast corner if at each turn he moves closer to his destination (assume all streets and avenues allow two-way traffic).

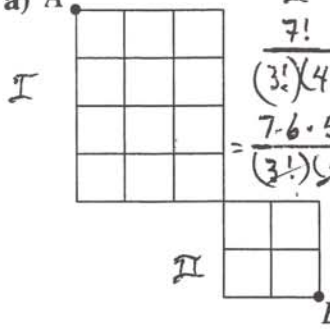
4N 5W $= \frac{9!}{4!\,5!}$

= 126 ways

Class Ex. #2

Find the number of pathways from A to B if paths must always move closer to B.

a) A



I $\quad \frac{7!}{(3!)(4!)} \cdot \frac{4!}{(2!)(2!)}$

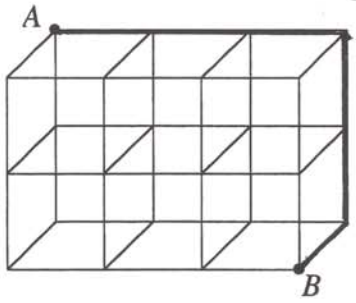
$= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{(3!)(4!)} \cdot \frac{4 \cdot 3 \cdot 2!}{(2!)(2!)}$

$= (35)(6)$

II

B $= 210$ pathways

b) A



B

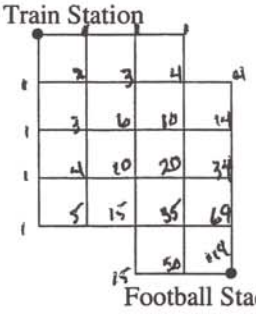
- One possible path is shown
- Any path from A to B must travel 3 Right 2 Down 1 Forward
- The number of pathways is $\frac{6!}{(3!)(2!)}$

$= 60$ pathways

Class Ex. #3

A taxi company is trying to find the quickest route during rush hour traffic from the train station to the football stadium. How many different routes must must be considered if at each intersection the taxi must always move closer to the football stadium?

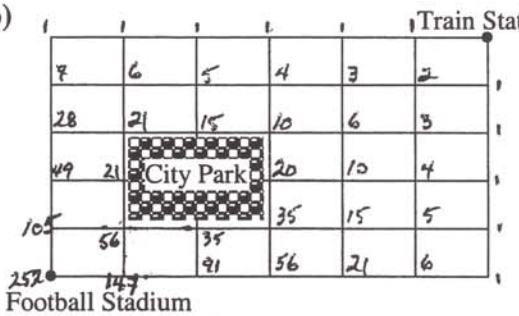
a) Train Station



Football Stadium

119 different routes

b) Train Station



Football Stadium

252 different routes

Class Ex. #4

How many different paths can a pinball take as it falls from top to bottom?

Pascal

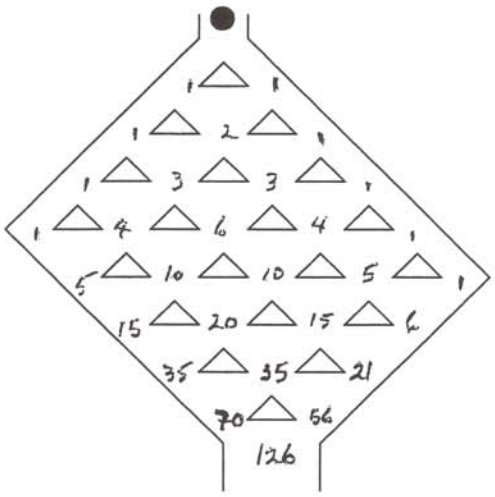
126 different paths

or $\frac{9!}{5!4!}$

$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5! 4!}$

$= \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$

$= 126$ paths

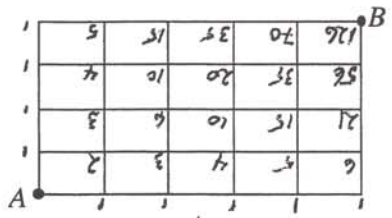


Complete Assignment Questions #1 - #7

Assignment

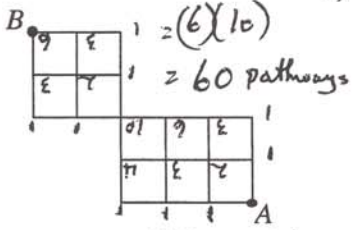
1. Find the number of pathways from A to B if paths must always move closer to B.

a)



or $\frac{9!}{5!4!} = 126$ pathways

b)



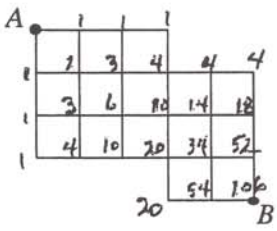
$= (6)(10)$

$= 60$ pathways

or $\left(\frac{4!}{2!2!}\right)\left(\frac{5!}{3!2!}\right)$

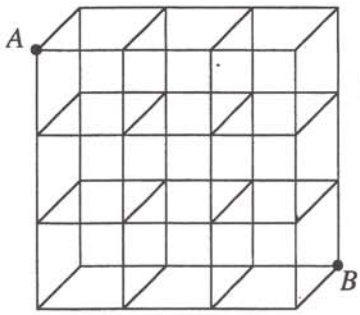
$= 60$ pathways

c)



106 pathways

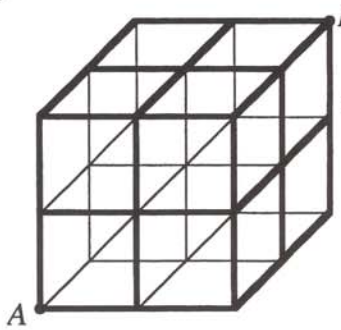
d)



$= \frac{7!}{3!3!}$

$= 140$ pathways

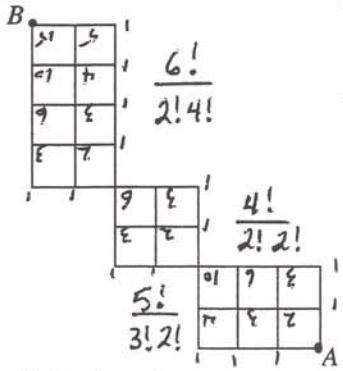
e)



$= \frac{6!}{2!2!2!}$

$= 90$ pathways

f)

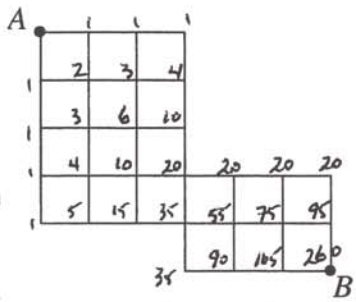


$\frac{6!}{2!4!}$ $\frac{4!}{2!2!}$ $\frac{5!}{3!2!}$

$= (10)(6)(15)$

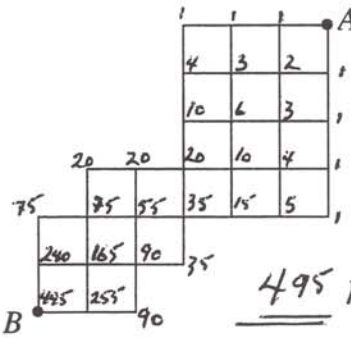
$= 900$ pathways

g)



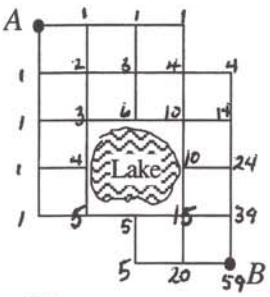
260 pathways

h)



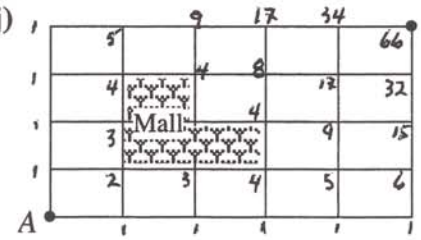
495 pathways

i)



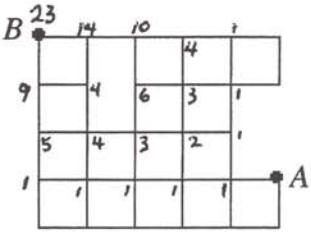
59 pathways

j)



66 pathways

k)



23 pathways

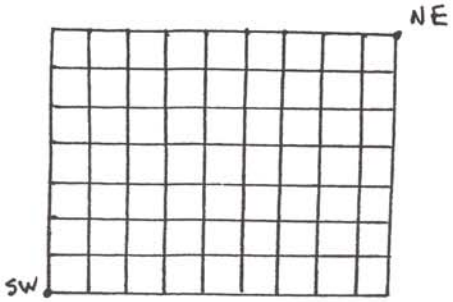
S ↓

2. The bakery is four blocks south and three blocks west of the supermarket. The bakery driver, bored with travelling the same route, decides to use a different route for each delivery. Assuming that he always travels closer to the supermarket, how many deliveries are possible before he has to repeat a route?

$= \frac{7!}{4!\,3!}$

$= 35$ different deliveries

3. A town has 10 streets running from north to south and 8 avenues running from west to east. A man wishes to drive from the extreme south-west intersection to the extreme north-east intersection, moving either north or east along one of the streets or avenues. Find the number of routes he can take. State any assumptions that you have made.



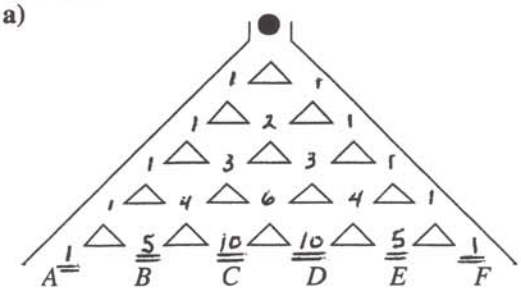
$= \frac{16!}{(9!)(7!)}$

$= 11\,440$ routes

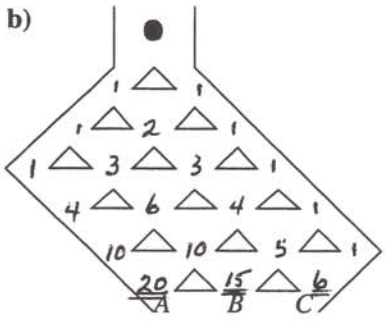
- Must drive closer to his destination
- None of the routes are one way

4. Given the following pinball machines determine the number of pathways to reach each of the exits.

a)



b)



Multiple Choice

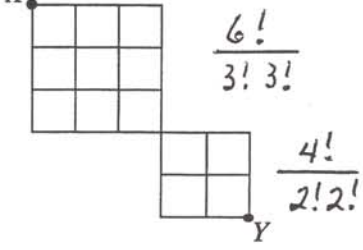
5. The number of pathways from X to Y if paths must always move closer to Y is

A. $\frac{6!}{3!\,3!} + \frac{4!}{2!\,2!}$

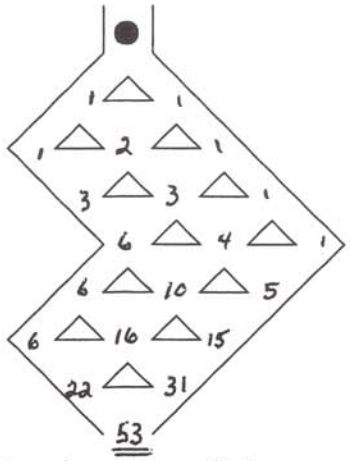
B. $\frac{6!}{3!\,3!} \times \frac{4!}{2!\,2!}$

C. $\frac{8!}{4!\,4!} + \frac{6!}{3!\,3!}$

D. $\frac{8!}{4!\,4!} \times \frac{6!}{3!\,3!}$



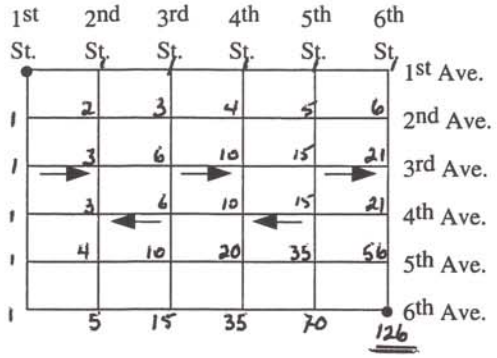
Numerical Response **6.** The number of different paths that a pinball can take as it falls from top to bottom is 53 .



(Record your answer in the numerical response box from left to right)

5	3		

7. Counsellor McAdam has to drive from ciy hall (at the corner of 1st Ave. and 1st St.) to the hospital (at the corner of 6th Ave. and 6th St.). Third Avenue is one way heading east and Fourth Avenue is one way heading west. If he must always be travelling closer to the hospital, the number of different routes he can take is 126 .



(Record your answer in the numerical response box from left to right)

1	2	6	

Answer Key

1. **a)** 126 **b)** 60 **c)** 106 **d)** 140 **e)** 90
f) 900 **g)** 260 **h)** 495 **i)** 59 **j)** 66 **k)** 23

2. 35 **3.** 11 440. Assumptions: 1) He always travels closer to his destination.
2) None of the routes are one-way from north to south or east to west

4. **a)** Exit A: 1 Exit B: 5 Exit C: 10 Exit D: 10 Exit E: 5 Exit F: 1
b) Exit A: 20 Exit B: 15 Exit C: 6

5. B **6.**

5	3		

7.

1	2	6	

Permutations and Combinations Lesson #8:
Probability Involving Permutations and Combinations

Warm-Up *Using Permutations or Combinations to Find the Probability of an Event*

Two cards are picked without replacement from a deck of 52 playing cards. Determine the probability that both are kings using

a) the multiplication law

$P(K_1 \text{ and } K_2) = P(K_1) \cdot P(K_2/K_1)$

$P(K_1 \text{ and } K_2) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right) = \frac{1}{221}$

b) combinations

$P(K_1 \text{ and } K_2) = \frac{\text{favorable outcomes}}{\text{total possible outcomes}}$

$P(K_1 \text{ and } K_2) = \frac{{}_4C_2}{{}_{52}C_2} = \frac{6}{1326} = \frac{1}{221}$

Class Ex. #1

The word **COUNTED** has been spelled using Scrabble tiles. Two tiles are randomly chosen one at a time and placed in the order in which they were chosen. Determine the probability that the tiles are:

a) CO

$P(C \text{ and } O) = \left(\frac{1}{7}\right)\left(\frac{1}{6}\right) = \frac{1}{42}$

or $= \frac{1}{{}_7P_2} = \frac{1}{42}$

b) both vowels

$P(V_1 \text{ and } V_2) = \left(\frac{3}{7}\right)\left(\frac{2}{6}\right) = \frac{6}{42} = \frac{1}{7}$

or $= \frac{{}_3P_2}{{}_7P_2} = \frac{6}{42} = \frac{1}{7}$

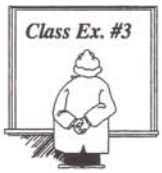
Class Ex. #2

The Athletic Council decides to form a sub-committee of seven council members to look at how funds raised should be spent on sports activities at the school. There are a total of 15 athletic council members, 9 males and 6 females. What is the probability that the sub-committee will consist of exactly 3 females?

$P(3F) = \frac{\#\text{ of ways to choose exactly three females}}{\#\text{ of ways to choose the sub-committee}}$

$= \frac{{}_6C_3 \cdot {}_9C_4}{{}_{15}C_7} = \frac{56}{143}$

Class Ex. #3



A bag of marbles contains 5 red, 3 green, and 6 blue marbles. If a child grabs three marbles from the bag, determine the probability that:

a) exactly 2 are blue

$P(2B) = P(2B \text{ and } 1 \text{ other})$

$= \frac{\#\text{ of ways to choose 2 blue and 1 other}}{\#\text{ of ways to choose any two marbles}}$

$= \frac{{}_6C_2 \cdot {}_8C_1}{{}_{14}C_3} = \frac{30}{91}$

b) at least one is blue $= P(\text{at least one is blue})$

$= P(1B \text{ \& } 2\text{other}) \text{ or } P(2B \text{ \& } 1\text{other}) \text{ or } P(3B \text{ \& } 0\text{others})$

$= \frac{{}_6C_1 \cdot {}_8C_2 + {}_6C_2 \cdot {}_8C_1 + {}_6C_3 \cdot {}_8C_0}{{}_{14}C_3} = \frac{11}{13}$

or $P(1 - \text{none are blue}) = 1 - \frac{{}_6C_0 \cdot {}_8C_3}{{}_{14}C_3} = \frac{11}{13}$

c) the first is red, the second is green and the third is blue

$P(1R \text{ and } 1G \text{ and } 1B)$ *order important*

$= \left(\frac{5}{14}\right)\left(\frac{3}{13}\right)\left(\frac{6}{12}\right) = \frac{15}{364}$

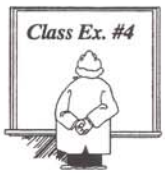
or $= \frac{{}_5P_1 \cdot {}_3P_1 \cdot {}_6P_1}{{}_{14}P_3} = \frac{15}{364}$

d) one is red, one is green and one is blue

$= P(1R \text{ and } 1G \text{ and } 1B)$ *order NOT important*

$= 3! \times \left(\frac{5}{14}\right)\left(\frac{3}{13}\right)\left(\frac{6}{12}\right) = \frac{45}{182}$

or $= \frac{{}_5C_1 \cdot {}_3C_1 \cdot {}_6C_1}{{}_{14}C_3} = \frac{45}{182}$



City Counicl consists of nine men and six women. Three representatives are chosen at random to form a environmental sub-committee.

a) What is the probability that Mayor Milonovich and two women are chosen?

MM → Mayor Milonovich M → men W → women

1MM 0M 2W

$$P(\text{MM and 2W}) = \frac{{}_1C_1 \cdot {}_8C_0 \cdot {}_6C_2}{{}_{15}C_3} = \frac{(1)(1)(15)}{455} = \frac{15}{455} = \frac{3}{91}$$

b) What is the probability that two women are chosen if Mayor Milonovich must be on the committee?

- Given Mayor Milonovich MUST be on the committee we assume he is already on the committee and ∴ not a consideration.
- Thus we are really choosing from 8 men NOT 9, and hence a total of 14 people NOT 15.
- Also, because MM must be on the committee of 3, we are really choosing the remaining positions on the committee, namely a group of 2.

$$P(2W) = \frac{{}_8C_0 \cdot {}_6C_2}{{}_{14}C_2} = \frac{(1)(15)}{91} = \frac{15}{91}$$

In a card game you are dealt 5 cards from a pack of 52 shuffled cards.
When you look at your 5 cards, what is the probability, expressed in combination notation, that you have:

a) four aces?

$P(4A) = P(4A \text{ and } 1 \text{ other})$

$$= \frac{{}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5}$$

b) four tens and an ace?

$P(4 \text{ tens and } 1A)$

$$= \frac{{}_4C_4 \cdot {}_4C_1}{{}_{52}C_5}$$

c) 10, J, Q, K and ace?

$P(10 \,\&\, J \,\&\, Q \,\&\, K \,\&\, A)$

$$= \frac{{}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1}{{}_{52}C_5}$$

$$= \frac{({}_4C_1)^5}{{}_{52}C_5}$$

d) at least one Jack?

$= P(1J \,\&\, 4 \text{ ot}) \text{ or } P(2J \,\&\, 3 \text{ ot}) \text{ or } P(3J \,\&\, 2 \text{ ot}) \text{ or } P(4J \,\&\, 1 \text{ ot})$

$$= \frac{{}_4C_1 \cdot {}_{48}C_4}{{}_{52}C_5} + \frac{{}_4C_2 \cdot {}_{48}C_3}{{}_{52}C_5} + \frac{{}_4C_3 \cdot {}_{48}C_2}{{}_{52}C_5} + \frac{{}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5}$$

$$= \frac{{}_4C_1 \cdot {}_{48}C_4 + {}_4C_2 \cdot {}_{48}C_3 + {}_4C_3 \cdot {}_{48}C_2 + {}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5}$$

OR

$$= 1 - P(\text{no Jacks}) = 1 - \frac{{}_4C_0 \cdot {}_{48}C_5}{{}_{52}C_5} = 1 - \frac{{}_{48}C_5}{{}_{52}C_5}$$

Complete Assignment Questions #1 - #13

Assignment

1. A marble is drawn at random from a box containing 10 red, 30 yellow, 20 blue and 10 pink marbles. Find, as an exact fraction, the probability that the marble drawn is:

a) yellow or red

$P(Y) + P(R)$

$= \frac{30}{70} + \frac{10}{70}$

$= \frac{4}{7}$

b) not blue

$1 - P(B)$

$= 1 - \frac{20}{70}$

$= \frac{5}{7}$

c) green

$P(G)$

$= \frac{0}{7}$

$= 0$

d) red, pink or blue

$P(R) + P(Pi) + P(B)$

$= \frac{10}{70} + \frac{10}{70} + \frac{20}{70}$

$= \frac{4}{7}$

2. A bag contains 6 blue marbles and 10 yellow marbles. Two marbles are drawn from the bag without replacement.

a) Use the multiplication law to determine:

i) *P*(both are blue)

$P(2 \text{ blue}) = \frac{6}{16} \times \frac{5}{15} = \frac{1}{8}$

ii) *P*(the first is blue and the second is yellow)

$P(1 \text{ blue then } 1 \text{ yellow}) = \frac{6}{16} \times \frac{10}{15} = \frac{1}{4}$

iii) *P*(one is blue and one is yellow)

$P(1 \text{ B and } 1 \text{ Y})$

$= P(1 \text{ B and } 1 \text{ Y}) \text{ or } P(1 \text{ Y and } 1 \text{ B})$

$= \frac{6}{16} \times \frac{10}{15} + \frac{10}{16} \times \frac{6}{15} = \frac{1}{2}$

b) Use combinations or permutations to determine:

i) *P*(both are blue)

$P(2 \text{ blue}) = \frac{{}_6C_2}{{}_{16}C_2} = \frac{1}{8}$

ii) *P*(the first is blue and the second is yellow)

$P(1 \text{ blue then } 1 \text{ yellow}) = \frac{{}_6P_1 \cdot {}_{10}P_1}{{}_{16}P_2} = \frac{1}{4}$

iii) *P*(one is blue and one is yellow)

$P(1 \text{ B and } 1 \text{ Y}) = \frac{{}_6C_1 \cdot {}_{10}C_1}{{}_{16}C_2} = \frac{1}{2}$

3. Two marbles are drawn at random from a box containing 10 red, 20 yellow, 15 blue and 15 pink marbles. Find the probability, to the nearest thousandth, that

a) both marbles are red

$P(R \text{ and } R)$

$= \frac{{}_{10}C_2}{{}_{60}C_2} = 0.025$

b) the first marble is red and the second marble is yellow

$P(R \text{ and } Y)$ *order important*

$= \frac{10}{60} \cdot \frac{20}{59} = 0.056$ OR $\frac{{}_{10}P_1 \cdot {}_{20}P_1}{{}_{60}P_2} = 0.056$

c) neither marble is pink

$P(\text{no pink})$

$= \frac{{}_{45}C_2}{{}_{60}C_2} = 0.559$

d) both marbles are either blue or red

$= P(2B \text{ or } 2R \text{ or } 1B \text{ and } 1R \text{ or } 1R \text{ and } 1B)$

$= \frac{{}_{25}C_2 \cdot {}_{35}C_0}{{}_{60}C_2} = 0.169$

4. In a card game you are dealt 5 cards from a pack of 52 well shuffled cards. What is the probability, expressed in combination notation, that you have:

a) four Queens?

$P(4Q) = P(4Q \text{ and } 1 \text{ other})$

$$= \frac{{}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5}$$

b) one face card?

$P(1 \text{ face card and } 4 \text{ others})$

$$= \frac{{}_{12}C_1 \cdot {}_{40}C_4}{{}_{52}C_5}$$

c) A, 2, 3, 4 and 5?

$P(A \,\&\, 2 \,\&\, 3 \,\&\, 4 \,\&\, 5)$

$$= \frac{{}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1}{{}_{52}C_5}$$

$$= \frac{({}_4C_1)^5}{{}_{52}C_5}$$

d) at most one King?

$= P(0K \,\&\, 5 \text{ others}) \text{ or } P(1K \,\&\, 4 \text{ others})$

$$= \frac{{}_4C_0 \cdot {}_{48}C_5}{{}_{52}C_5} + \frac{{}_4C_1 \cdot {}_{48}C_4}{{}_{52}C_5}$$

$$= \frac{{}_{48}C_5 + {}_4C_1 \cdot {}_{48}C_4}{{}_{52}C_5}$$

e) $1 - P(00)$

$$= 1 - \frac{{}_{39}C_5}{{}_{52}C_5}$$

5. Three prizes are rewarded in a raffle during a halftime show at a school basketball game. Ben, Janelle, Jaimie, and 17 other students each have one ticket.

a) If the raffle has a first, second, and third prize, what is the probability, as an exact value, that Ben wins first prize, Janelle wins second prize, and Jamie wins third prize?

order important $\qquad P(Ben_1 \,\&\, Janelle_2 \,\&\, Jamie_3) = \dfrac{{}_1P_1 \cdot {}_1P_1 \cdot {}_1P_1}{{}_{20}P_3} = \dfrac{1}{6840}$

b) If the raffle has three identical prizes, what is the probability that Ben, Janelle, and Jamie win the prizes?

order NOT important $\qquad P(Ben \,\&\, Janelle \,\&\, Jamie) = \dfrac{{}_3C_3}{{}_{20}C_3} = \dfrac{1}{1140}$

6. Three toys are selected at random from a box of nine toys, three of which are defective. Find the probability (exact value) that of the toys selected:

a) none are defective

$P(3G \text{ and } 0D)$

$$= \frac{{}_6C_3 \cdot {}_3C_0}{{}_9C_3}$$

$$= \frac{5}{21}$$

b) there are more defective than non-defective

$P(3D \text{ and } 0G) \text{ or } P(2D \text{ and } 1G)$

$$= \frac{{}_3C_3 \cdot {}_6C_0}{{}_9C_3} + \frac{{}_3C_2 \cdot {}_6C_1}{{}_9C_3}$$

$$= \frac{1}{84} + \frac{18}{84} = \frac{19}{84}$$

7. A bank card personal identification number consists of any four digits. Repeat digits are allowed and the code can start with zero. What is the probability that a code begins and ends with 5?

# of ways to choose 4 digit #'s which begins & ends with 5: $1 \times 10 \times 10 \times 1$

# of ways to choose four digit numbers: $10 \times 10 \times 10 \times 10$

$$P(\text{code}) = \frac{\#\text{ of ways to choose 4 digit \#'s which begins \& ends with five}}{\#\text{ of ways to choose 4 digit numbers}}$$

$$= \frac{100}{10\,000} = \frac{1}{100}$$

8. In a small rural town there is only one football field to be shared by the five local teams. In the league competition, each team plays two games against every other team. What is the probability that the first game scheduled will have last year's two best teams, Oilmen and Wheatmen in opposition?

# of ways Oilmen and Wheatmen play is 2

If Each team played each other only once, then the total # of games would be ${}_5C_2 = 10$. But each team plays each other twice, so the total # of games is $2 \times {}_5C_2 = 20$.

$$P(O \ \& \ W) = \frac{2}{2 \times {}_5C_2} = \frac{1}{10}$$

9. A jar contains 32 balls, equal numbers of red, blue, green and yellow. Five balls are drawn from the jar, simultaneously and at random. Determine the probability (to the nearest thousandth) of each event

a) There are three red balls and two yellow ones

$$P(3R \text{ and } 2Y) = \frac{{}_8C_3 \cdot {}_8C_2 \cdot {}_{16}C_0}{{}_{32}C_5} = 0.008$$

b) The balls are three of one colour and two of another colour

# of ways to choose a colour ${}_4P_2 = 12$.

# of ways to choose three balls with same colour is ${}_8C_3 = 56$.

# of ways to choose two balls with same colour is ${}_8C_2 = 28$.

$$P(3 \text{ one col and 2 ano col}) = \frac{{}_4P_2 \cdot {}_8C_3 \cdot {}_8C_2}{{}_{32}C_5} = 0.093$$

c) All five balls have the same colour

$$P(\text{same colour}) = P(5R) \text{ or } P(5B) \text{ or } P(5G) \text{ or } P(5Y) = \frac{{}_8C_5}{{}_{32}C_5} + \frac{{}_8C_5}{{}_{32}C_5} + \frac{{}_8C_5}{{}_{32}C_5} + \frac{{}_8C_5}{{}_{32}C_5} = 0.001$$

$$\text{or} \quad = \frac{{}_4C_1 \cdot {}_8C_5}{{}_{32}C_5} = 0.001$$

10. There are twelve boys and ten girls in an English 30 class. A yearbook committee of four is chosen at random.

a) What is the probability, to the nearest hundredth, that Ryan, the school president, and three girls are on the committee?

$R \rightarrow$ Ryan $\quad B \rightarrow$ boys $\quad G \rightarrow$ girls

$$P(R \text{ and } 2G) = \frac{\overset{1R}{{}_1C_1} \cdot \overset{0B}{{}_{11}C_0} \cdot \overset{2G}{{}_{10}C_3}}{{}_{22}C_4} = \frac{(1)(1)(120)}{7315} = \frac{120}{7315} = \frac{24}{1463} = 0.02$$

b) If Ryan must be on the committee, what is the probability, to the nearest hundredth, that three girls are chosen?

- Given Ryan MUST be on the committee we assume he is already on the committee and $\therefore$ not a consideration.
- Thus we are really choosing from 11 boys NOT 12, and hence a total of 21 people NOT 22.
- Also, because R must be on the committee of 4, we are really choosing the remaining positions on the committee, namely a group of 3.

$$P(2G) = \frac{{}_{11}C_0 \cdot {}_{10}C_3}{{}_{21}C_3} = \frac{(1)(120)}{1330} = \frac{120}{133} = 0.09$$

11. A six digit numeral is represented by the digits 1, 2, 3, 4, 5 and 6 in any order. If one of these numerals is chosen at random what is the probability that:

a) all the digits are in decreasing order?

$$P(\text{dig dec}) = \frac{\text{favourable outcomes}}{\text{total outcomes}} = \frac{1}{6!} \text{ or } \frac{1}{{}_6P_6} = \frac{1}{720}$$

b) the even digits are in decreasing order?

$$P(\text{even dig}) = \frac{\text{favourable outcomes}}{\text{total outcomes}} = \frac{1}{3!} \text{ or } \frac{1}{{}_3P_3} = \frac{1}{6}$$

Multiple Choice **12.** Chandra buys two of a total of thirty raffle tickets. There are two winning numbers. The probability that Chandra wins exactly one prize is

A. $\frac{56}{435}$ **B.** $\frac{28}{435}$ **C.** $\frac{2}{435}$ **D.** $\frac{1}{435}$

$$P(\text{chandra wins exactly one prize}) = \frac{\text{favourable outcomes}}{\text{total outcomes}} = \frac{{}_2C_1 \cdot {}_{28}C_1}{{}_{30}C_2} = \frac{56}{435}$$

Numerical Response **13.** The letters of the word **OKOTOKS** are arranged. The probability, to the nearest hundredth, that an arrangement, chosen at random from all possible arrangements, begins and ends with a vowel is _____ .

(Record your answer in the numerical response box from left to right)

# of ways to choose an arrangement which begins and ends with a vowel. $= \frac{3 \cdot 2 \cdot 5!}{3! \cdot 2!} = 60$

# of ways to choose any 7 letter arrangement. $= \frac{7!}{3!\,2!} = 420$

$P(\text{begins and ends with a vowel}) = \frac{\text{favourable outcomes}}{\text{total outcomes}} = \frac{60}{420} = \frac{1}{7} = 0.14$

Answer Key

1. a) $\frac{4}{7}$ **b)** $\frac{5}{7}$ **c)** 0 **d)** $\frac{4}{7}$ **2. i)** $\frac{1}{8}$ **ii)** $\frac{1}{4}$ **iii)** $\frac{1}{2}$

3. a) 0.025 **b)** 0.056 **c)** 0.559 **d)** 0.169

4. a) $\frac{{}_4C_4\,{}_{48}C_1}{{}_{52}C_5}$ **b)** $\frac{{}_{12}C_1\,{}_{40}C_4}{{}_{52}C_5}$ **c)** $\frac{({}_4C_1)^5}{{}_{52}C_5}$ **d)** $\frac{{}_4C_0\,{}_{48}C_5}{{}_{52}C_5} + \frac{{}_4C_1\,{}_{48}C_4}{{}_{52}C_5}$ **e)** $1 - \frac{{}_{39}C_5}{{}_{52}C_5}$

5. a) $\frac{1}{6840}$ **b)** $\frac{1}{1140}$ **6. a)** $\frac{5}{21}$ **b)** $\frac{19}{84}$ **7.** $\frac{1}{100}$ **8.** $\frac{1}{10}$

9. a) 0.008 **b)** 0.093 **c)** 0.001

10. a) 0.02 **b)** 0.09

11. a) $\frac{1}{720}$ **b)** $\frac{1}{6}$ **12.** A **13.**

0	.	1	4